The Basics of Ramsey Theory

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0.1 The Ramsey Theorem

Let K_N denote the complete graph on N vertices. Suppose that the edges of K_N have been colored red and blue. We say that a *red* K_m is a collection of m vertices such that every edge between these vertices is colored red. Likewise define a *blue* K_n. Here is the Ramsey Theorem.

Theorem 0.1 *Let $m, n \geq 2$ be integers. There exists some integer $R(m, n)$ such that if $N \geq R(m, n)$ and R_N has any red-blue edge coloring, then there exists a red K_m or a blue K_n.*

Proof: The proof goes by induction on $m + n$. The base case is $m = n = 2$. In this case we have $R(2, 2) = 2$ because this single edge must be colored either red or blue. Suppose by induction that the result is true for $R(m-1, n)$ and $R(m, n-1)$. Define $M = \max\ R(m-1, n), R(m, n-1)$. Consider any red-blue edge coloring of K_{2M}. Call this graph G.

We pick some vertex v of K_{2M}, This vertex has degree $2M - 1$. Hence, there must be at least M edges incident to v which have the same color. Let us say that this color is red. (The blue case has the same treatment, except that we work with the pair $(m, n-1)$ rather than the pair $(m-1, n)$.)

We let G' be the subgraph induced by all the vertices of G connected to v by red edges. The graph G' has a red-blue edge-coloring, and G' has at least $M \geq K(m-1, n)$ vertices. Therefore, by induction, G' either has a red K_{m-1} or a blue K_n. If G' has a blue K_n we are done. If G' has a red K_{m-1}, then $G' \cup v$ is colored in such a way as to have a red K_m because every edge connecting v to a vertex of G' is red.

This completes the induction step. ♠

0.2 The Ramsey Hypergraph Theorem

A *k-hypergraph* is a collection V of vertices together with a subset of k-element subsets of V. A 2-hypergraph is just a graph. One can think of a 3-hypergraph as a collection of triangles whose vertices are in K. The k-hypergraphs for larger values of k have similar geometric interpretations in terms of higher dimensional tetrahedra – i.e., simplices.

The *complete k-hypergraph* K_N^k is the k-hypergraph whose vertex set is $V_N = \{1, ..., N\}$ and whose "edges" are the set of all possible k-element subsets of V. A red-blue coloring K_N^k is an assignment of one of the two colors, red or blue, to each of the k-element subsets of V_N. Just as for graphs, a *red* K_n^k is a subset V_n' of n vertices of V_N such that every k-element subset of V_n' is colored red. We define a *blue* K_n^k in the same way.

Here is the general Ramsey Hypergraph Theorem.

Theorem 0.2 *Given m, n, k there is some number $R(m, n; k)$ with the following property. If $N \geq R(m, n; k)$, then every red-blue coloring of K_N^k either has a red K_m^k or a blue K_n^k.*

Proof: The proof is a double induction argument. We have already proved the result for $k = 2$. Suppose then that we have the smallest value of k for which we do not know the result. We need at least k vertices to have a nontrivial k-hypergraph. We can see that $R(k, n; k) = n$ for $n \geq k$ because every edge coloring of K_n^k is either entirely blue or has at least one red "edge".

With k fixed, proof now goes by induction on $m + n$. Choose

$$N = 1 + R(M, M, k-1), \qquad M = \max R(m-1, n, k), R(m, n-1, k).$$

Consider G_N^k. Choose some vertex v of G_N^k. We can form the auxiliary complete $(k-1)$ hypergraph on the vertices of $G_N^k - v$. An edge of this auxiliary hypergraph is some collection $\{w_1, ..., w_{k-1}\}$ of vertices of G_N^k which do not contain v. We color this collection red or blue according as the collection $\{w_1, ..., w_{k-1}, v\}$ is colored red or blue.

By induction, there is either a red K_M^{k-1} or a blue K_M^{k-1} in this auxiliary graph. Assume w.l.o.g. the red option occurs. We can throw out some vertices of our original k-hypergraph and get a new one, G_M^k, which is colored so that that every k-element subset containing v is red. By induction this thing either contains a red G_{m-1}^k or a blue G_n^k. In the latter case, we are done. in the former case, we get our red G_m^k in the graph $G_{m-1}^k \cup v$. ♠

0.3 Application to Convex Geometry

A subset S of the plane is *convex* if it has the following property: if any two points lie in S then so does the line segment connecting them. The intersection of any number of convex sets is either empty or convex. Given any bounded subset S in the plane, let $H(S)$ denote the intersection of all c convex sets which contain S. This intersection is called the *convex hull* of S. It is the smallest convex set containing S.

Say that a finite set S of points in the plane is *point convex* if every point in S lies on the boundary of $H(S)$. For instance, the set of vertices of a square is point convex. In general, a set of points is point convex if and only if it is the set of vertices of a convex polygon. Say that a collection of points in the plane is int *general position* if no three of them lie on the same line. Here is an application of Ramsey's 4-hypergraph theorem.

Theorem 0.3 *Given any positive integer n there is some N such that any collection of N general position points contains a point convex subset with n elements.*

Proof: We first note that any 5-element subset of points contains a 4-element subset which is vertex convex. (Draw a few pictures and you will become convinced.) We choose $n \geq 5$ and $N = R(n, n, 4)$. We color each 4-element subset of our points blue if it is vertex convex and red if not. By the Ramsey theorem, we have either a red K_n^4 or a blue K_n^4. But we have just gotten through saying that we cannot have a red K_n^4. Hence we have a blue K_n^4. That is, we have n points such that every 4-element subset is vertex convex.

We claim that such a set S is vertex convex. If not, then there is vertex $v \in S$ which lies in the interior of the convex hull $H(S)$. But then there are 3 other vertices $w_1, w_2, w_3 \in S$ such that v lies in the triangle $\Delta(w_1, w_2, w_3)$. But then v, w_1, w_2, w_3 is not a vertex-convex 4-element subset. This is a contradiction. Hence S is vertex convex. ♠

There is a second proof in which we can deduce the same result from the Ramsey 3-hypergraph theorem. Label the points with numbers $1, ..., N$. Color each 3 element subset (i, j, k) red if three points v_i, v_j, v_k are clockwise oriented. Otherwise color them blue. Here we take $N = R(n, n, 3)$. By the Ramsey Hypergraph Theorem, there is either a red K_n^3 or a blue K_n^3. w.l.o.g. assume that there is a red K_n^3. Then we have n labeled points such that every triple is labeled clockwise. Call this set S.

We claim that this forces S to be vertex convex and labeled so that they go in order around the convex hull. The proof goes by induction on n. It is certianly true for $n = 3$, and an easy case by case analysis shows that it is also true for $n = 4$. But then every 4-element subset of S is vertex convex. Hence, as in the proof above, S itself is vertex convex.

The same theorem generalizes to higher dimensions, though the proof is harder. I think that actually the second proof, using orientation, is easier.

0.4 The Erdos Szekeres Theorem

Let us give another application of Ramsey theory.

Lemma 0.4 *Given any n there is some N such that any length N sequence of distinct integers has a monotone (either increasing or decreasing) length n subsequence.*

Proof: Call the sequence $d_1, ..., d_N$. Color each pair (i, j) red or blue according as $(i - j)(d_i - d_j)$ is positive or negative. By the Ramsey Theorem, there is either a red K_n or a blue K_n. The former corresponds to a decreasing length n subsequence and the latter corresponds to an increasing length n subsequence. ♠

This is a nice application but the Erdos-Szekeres Theorem says that one can take $N = (n - 1)^2 + 1$ in the above result. This is much better than what you would get from using the Ramsey numbers. Here is one proof. (I learned this proof on wikipedia.) We'll suppose this result false and derive a contradiction. For each i let a_i denote the length of the longest monotone increasing subsequence that ends with the number d_i. Likewise define b_i for decreasing. Call (a_i, b_i) a *tag*. If $i < j$ then $(a_i, b_i) \neq (a_j, b_j)$ because of the following:

- if $d_j > d_i$ then $a_j \geq a_i + 1$. The point here is that the maximal increasing subsequence ending at d_i can be extended to d_j.
- if $d_j < d_i$ then $b_j \geq b_i + 1$. Similar reason.

In short, all the tags are distinct. If $a_i \in \{1, ..., n - 1\}$ and $b_i \in \{1, ..., n - 1\}$ for all i then there are only $(n-1)^2$ possible tags. But there is one additional tag and so the assumption we just made is impossible.