A Criterion for Hausdorff Quotients

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The purpose of these notes is to give more details about the result I mentioned in class about quotients being Hausdorff. These notes are more loquacious than the previous version.

1 Local Compactness

Recall that a topological space X is *Hausdorff* if for all distinct $p, q \in X$ there are open sets U_p and U_q such that $p \in U_p$ and $q \in U_q$ and $U_p \cap U_q = \emptyset$. The way this is commonly said is that we can separate points by open sets in a Hausdorff space.

Assume that X is Hausdorff. The space X is *locally compact* if for each $p \in X$ there are sets U_p and K_p such that $p \in U_p \subset K_p$, and U_p is open and K_p is compact. Any compact Hausdorff space is, of course, locally compact. But, there are lots of non-compact examples as well. For instance, Euclidean space $\boldsymbol{R}^n$ is locally compact. More generally, any closed subset of $\boldsymbol{R}^n$ is locally compact.

Here is an example of a space that is not locally compact. First consider $\boldsymbol{Z}$ (the integers) with the discrete topology. Then let X be the cone on $\boldsymbol{Z}$. You can picture X as a "prong", with an infinite number of edges sticking out of a common point. There are plenty of compact sets containing the prong point, but you can't arrange for one of them to also contain an open set. That is, X is not locally compact. Note that any other point of X does satisfy the criterion, because any other point of X has a neighborhood which is homeomorphic to $\boldsymbol{R}$.

As another example, equip $\boldsymbol{Q}$ (the rationals) with the subspace topology. $\boldsymbol{Q}$ is not locally compact.

2 Proper Discontinuity

Suppose that X is a compact Hausdorff space. Let G be group action on X. Technically this means that there is a continuous map $F : G \times X \to X$ such that

$$F(g, F(h,x)) = F(gh, x).$$

This way of writing things is annoying. The more usual way is to introduce the notation

$$g(x) = F(g, z).$$

In this way, we think of g acting on X as a homeomorphism for each $g \in X$. The compatibility condition is then

$$g(h(x)) = (gh)(x).$$

The group action is *properly discontinuous* if, for each compact $K \subset X$, the set

$$\{g \in G|\ g(K) \cap K \neq \emptyset\}$$

is finite.

Here are some examples:

- The usual action of $\boldsymbol{Z}$ on $\boldsymbol{R}$ is properly continuous. Here $n(x) = x + n$.
- More generally, the usual action of $\boldsymbol{Z}^n$ on $\boldsymbol{R}^n$ is properly discontinuous.
- Let
$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$
Let $\boldsymbol{G} = \boldsymbol{Z}$ and let $X = \boldsymbol{R}^2 - \{(0,0)\}$. The action is given by
$$n(v) = T^n(v).$$
This action is not properly discontinuous: If K is the unit circle centered at the origin, then $T^n(K)$ intersects K for all n.
- The action of $SL_2(\boldsymbol{Z})$ on the upper half plane discussed in class is properly discontinuous. Here $SL_2(\boldsymbol{Z})$, the group of integer 2×2 matrices of determinant 1, acts by linear fractional transformations.

3 The Main Result

Here is the main result.

Theorem 3.1 *Suppose that X is Hausdorff and locally compact, and G acts properly discontinuously on X. Then X/G is Hausdorff.*

4 Characterizing Open Sets

The rest of the notes are devoted to the proof.

Let $\pi : X \to X/G$ denote the quotient map. Say that a *G-invariant* open set is an open set U such that $g(U) = U$ for all $g \in G$. If $V \subset X/G$ is an open set, then $\pi^{-1}(V)$ is G-invariant. Conversely, if $U \subset X$ is open and G-invariant, then $\pi(U)$ is open. The point of this last claim is that $U = \pi^{-1}(\pi(U))$ when U is G-invariant. So, in short, G-invariant open sets in X correspond to open sets in X/G.

To prove that X/G is Hausdorff, we have to be able to separate any two distinct points $[x], [y] \in X/G$ by open sets. Let $x, y \in X$ be any points in the equivalence classes of $[x]$ and $[y]$ respectively. Separating $[x]$ and $[y]$ by open sets in X/G is the same as separating x and y by G-invariant open sets in X. The next result shows that we can get away with a little less.

Lemma 4.1 *Suppose we can find an open set V and a G-invariant open set W such that $V \cap W = \emptyset$, and $x \in V$ and $y \in W$. Then $[x]$ and $[y]$ are separated by open sets in X/G.*

Proof: Since V is disjoint from W and each $g \in G$ gives a homeomorphism of X preserving W, we see that $g(V)$ is disjoint from $g(W) = W$. But then $V' = G(V)$ is disjoint from W. But then V' and W are disjoint G-invariant open sets containing x and y respectively. ♠

The lemma tells us that it suffices to put any open set around x and then find a disjoint G-invariant open set around y. The local compactness lets us find "small" open sets around x and y. The proper discontinuity lets us arrange that the small open set around x is disjoint from all but finitely many images in the G-orbit of the small open set around Y. Then, using the Hausdorff property of X, we can shrink our small open sets so as to fix up the finitely many overlaps left over from the initial construction. The next section carries out the details.

5 The Details

Since X is locally compact, there are sets U_x, K_x and U_y, K_y where

- $x \in U_x \subset K_x$
- $y \in U_y \subset K_y$
- U_x, U_y are open
- K_x, K_y are compact.

From the proper discontinuity property there are only finitely many $g \in G$ such that

$$K_x \cap g(K_y) \neq \emptyset.$$

(You get this by applying the basic definition to $K = K_x \cup K_y$.) Call these exceptional elements $g_1, ..., g_n$.

As this point we just have finitely many overlaps to worry about. Now we explain the final shrinking argument.

Lemma 5.1 *For each $j = 1, ..., n$ we can find open sets V_j and W_j such that $x \in V_j \subset U_x$ and $y \in W_j \subset U_y$ and $V_j \cap g_j(W_j) = \emptyset$.*

Proof: Fix j. Since X is Hausdorff, we can find open sets V_j' and W_j' so that $x \in V_j'$ and $g_j(y) \in W_j'$ and $V_j' \cap W_j' = \emptyset$. We set $V_j = V_j' \cap U_x$ and

$$W_j = g_j^{-1}(W_j') \cap U_y.$$

These sets do the trick. ♠

Finally, define

$$V = \bigcap_{j=1}^{n} V_j, \qquad W = \bigcap_{j=1}^{n} W_j.$$

Note that $x \in V$ and $y \in W$. By construction V_j is disjoint from $g_j(W)$ for $j = 1, ..., n$. Moreover, for any other $g \in G$ we have

$$V \cap g(W) \subset K_x \cap g(K_y) = \emptyset.$$

Therefore V is disjoint from the G-invariant set $G(W)$. This completes the proof of the theorem.