Spring 2014 Math 1530 Final: Prof. Schwartz

Instructions: Pick 4 of the group problems and 4 of the ring problems and work on these. (So, you are working on 8 total.) You can use your book and your class notes for this exam, but not any other sources. The exam is due on **Tuesday 13 May at 3PM**. Please put your exam in a sealed envelope and slide the exam under my (Kassar 302) office door anytime before the due time. I may take off some points for sloppy presentation, so please write neatly.

G1. Recall that the regular dodecahedron is a platonic solid with 12 pentagonal faces. Let G be the group of orientation-preserving symmetries of the regular dodecahedron. Prove that G cannot have a normal subgroup of order 5.

G2. Let G be a group. Recall that the *commutator subgroup* of G is the subgroup generated by elements of the form $ghg^{-1}h^{-1}$, with $g,h \in G$. The commutator subgroup is denoted by $[G:G]$. Prove the following things:

- $[G:G]$ a normal subgroup
- The quotient $G/[G:G]$ is abelian.
- If N is normal subgroup of G, and G/N is abelian, then $[G:G] \subset N$.

G3. Let G be a group. Let $\mathcal{A}(G)$ denote the group of all automorphisms of G. Let $\mathcal{I}(G)$ denote the group of all inner automorphisms of G. Prove that $\mathcal{I}(G)$ is a normal subgroup of $\mathcal{A}(G)$.

G4. Let p be a prime. Find an order-p automorphism of the group $\boldsymbol{Z}/p \times \boldsymbol{Z}/p$ and use it to prove that there exists a non-abelian group of order p^3.

G5. (This is a problem from the book.) A group G is *solvable* if there is a sequence of subgroups

$$G_n \subset G_{n-1} \subset ... \subset G_0 = G$$

such that for all j, the subgroup G_{j+1} is normal in G_j, and the quotient G_j/G_{j+1} is abelian. Prove that all groups of order less than 60 are solvable.

R1. Let R be a commutative ring. An ideal I of R is *prime* if the statement $rs \in I$ always implies that either $r \in I$ or $s \in I$.

- Prove that I is a prime ideal of R if and only if R/I is an integral domain.
- Given an example of a ring R and an ideal I of R which is prime but not maximal.
- Give an example of a ring R and an ideal I such that I is maximal but R/I is not a field.

R2. Let R be a commutative ring and let $S \subset R$ be a subset. Let $R(S)$ be the set of all sums of the form

$$r_1 s_1 + ... + r_n s_n,$$

where $r_1, ..., r_n \in R$ and $s_1, ..., s_n \in S$ and $n \in \boldsymbol{N}$. Prove that $R(S)$ is an ideal of R and that any other ideal of R which contains S also contains $R(S)$. Thus, $R(S)$ is the smallest ideal containing S.

R3. Let n be a positive integer and let $n = p_1^{e_1} ... p_k^{e_k}$ be the prime factorization of m into positive primes. Here e_j tells how many times the prime p_j occurs as a factor. Prove that n can be written as the sum of two squares (of integers) if and only if e_j is even whenever p_j is congruent to 3 mod 4.

R4. Call a positive integer n *square-free* if n is not divisible by the square of any integer. For instance, 15 is square-free but 18 is not. Suppose n is a square-free integer and m is a positive integer greater than 1. Prove that there is no rational number r such that $r^m = n$. Put another way, you are supposed to prove that any mth root of n is irrational.

R5. Give examples of two countably infinite rings R_1 and R_2, both of which are integral domains of characteristic 2, which are not isomorphic to each other.