

Math 153 Final Assignment: Prof. Schwartz

Instructions: Do as many or as few problems as you like. Either turn it in or not. This doesn't count towards your grade. These problems are not indicative of the nature of the final exam for the class.

1. Let R be the ring of half-integral quaternions. That is $R = \mathbf{Z}[\zeta, i, j, k]$, where $\zeta = (1 + i + j + k)/2$. Let G denote the group of 24 units in R . The convex hull P of G is known as the 24-cell. Prove the following claims about P :

- P has 24 faces, all regular octahedra.
- The convex hull of the centers of the faces of P is a smaller copy of P .
- The symmetry group of P has order 1152.
- One can tile \mathbf{R}^4 by translated copies of P .

So, basically, fill in the details of the notes I handed out about the 4-square theorem.

2. Recall that we have the spin cover $\phi : SU(2) \rightarrow SO(3)$, where $SU(2)$ is the group of unit-norm quaternions and $SO(3)$ is the group of rotations of the 2-sphere. Let $\Gamma' \subset SO(3)$ denote the rotation symmetry group of the icosahedron and let $\Gamma = \phi^{-1}(\Gamma')$. The group Γ is a group of 120 unit-norm quaternions, called the *binary icosahedral group*. Prove that the convex hull of the points in Γ (considered as a subset of \mathbf{R}^4) is a 600-sided polytope with faces which are regular tetrahedra.

3. Suppose that you like the results of Problems 1 and 2 but you don't want to work out the details by hand. Explain as precisely as possible how you *could* verify all the assertions in these two problems with the aid of a computer. Better yet, *write* a computer program which does this.

4. An automorphism of a graph is a bijection between the vertices which maps edges to edges. Let G be an arbitrary finite group. Prove that there is some graph Γ_G such that G is isomorphic to the automorphism group of Γ_G .

5. Let p and q be distinct primes. Prove that the ring of p -adic integers is not isomorphic to the ring of q -adic integers.
6. Prove that there are uncountably many (pairwise non-isomorphic) countable integrable domains of characteristic 2.
7. Let S_n denote the permutation group of n things, and suppose that $n > 6$. Prove that every automorphism of S_n is an inner automorphism.
8. Let $\omega = \exp(2\pi i/n)$. Consider the ring $R = \mathbf{Z}[\omega]$. A typical element of R has the form $\sum a_i \omega^i$. Prove that R is a dense subset of \mathbf{C} if $n > 6$.