Math 153 Midterm 1 Solutions: Prof. Schwartz

1. (a) To show that $HK \subset KH$ let $hk \in HK$. Since K is normal, $hkh^{-1} \in K$. Hence $hk \in Kh \subset KH$. This shows that $HK \subset KH$. The same argument, with the roles of H and K reversed, shows that $KH \subset HK$. Hence $HK = KH$.

(b) Let $hk \in HK$. Then $(hk)^{-1} = k^{-1}h^{-1} \subset KH = HK$. This shows that HK is closed under inverses. Also, since $HK = KH$, we have

$$h_1k_1h_2k_2 = h_1(k_1h_2)k_2 = h_1(h_3k_3)k_2 = h_4k_4 \in HK.$$

This shows that HK is closed under composition. Being closed under composition and inverses, HK is a subgroup.

(c) Let $g \in G$ be arbitrary. $gHK = HgK = HKg$. The first equality comes from the fact that the left and right cosets of H are equal. The second equality comes from the fact that the left and right cosets of K are equal.

2. Let $g \in G$ be some element. Suppose first that g has finite order n. Let p be a prime dividing n. Write $n = kp$. Then g^k has order p, and the group generated by g^k has prime order. If g has infinite order, define $\phi : \boldsymbol{Z} \to H$ by $\phi(n) = g^n$. By definition of H, the map ϕ is onto. Note that $\phi(m+n) = g^{n+m} = g^m g^n = \phi(m)\phi(n)$. This shows that ϕ is a homomorphism. Since g has infinite order, the kernel of ϕ is trivial. Hence ϕ is an isomorphism between $\boldsymbol{Z}$ and the subgroup generated by g.

3. (a) The easiest example is to take $G_1 = \boldsymbol{Z}/6$ and $G_2 = S_3$ the group of permutations of 3 things. Since G_1 is abelian and G_2 is not abelian, there is no isomorphism between G_1 and G_2.

(b) If D_4 and Q were isomorphic, they would have the same number of elements of order 4. But 6 elements of Q have order 4 whereas only 2 elements of D_4 have order 4. Hence, these groups are not isomorphic.