The Spin Cover

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The purpose of these notes is to describe the famous *spin cover*. This is the 2-to-1 surjective homomorphism from $SU(2)$ to $SO(3)$. Here $SU(2)$ is the group of unit quaternions and $SO(3)$ is the group of orientation preserving rotations of $\boldsymbol{R}^3$ which fix the origin.

1 The Group of Unit Quaternions

A *quaternion* is an object of the form

$$q = a + bi + cj + dk, \qquad a, b, c, d \in \boldsymbol{R}, \tag{1}$$

where the symbols i, j, k are subject to the rules that

- $i^2 = j^2 = k^2 = -1$.
- $ij = k$ and $jk = i$ and $ki = j$.
- $ji = -k$ and $kj = -i$ and $ik = -j$.

Quaternions are added and subtracted component by component. They are multiplied together using the distributive law together with the rules above. With these operations, the quaternions form a non-commutative ring. This fact is easy, but somewhat tedious, to check directly.

The *conjugate* of a quaternion q is given by the formula

$$\overline{q} = a - bi - cj - dk. \tag{2}$$

Here q is as in Equation 1. Here is how conjugation and multiplication interact.

Lemma 1.1 *For any two quaternions q and r, we have $\overline{qr} = (\overline{r})(\overline{q})$.*

Proof: You can check this directly for the 16 choices $q, r \in \{1, i, j, k\}$. For instance

$$\overline{i^2} = -1 = (-i)(-i) = (\bar{i})^2, \qquad \overline{ij} = -k = ji = (-j)(-i) = (\bar{j})(\bar{i}).$$

The general case follows from these special cases and the distributive law. ♠

The *norm* of a quaternion is given by

$$|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}. \tag{3}$$

Lemma 1.2 *For any two quaternions q and r, we have $|qr| = |q||r|$.*

Proof: We have

$$|qr|^2 = (qr)(\overline{qr}) = qr\overline{r}\overline{q} = q|r|^2\overline{q} =_* q\overline{q}|r|^2 = |q|^2|r|^2.$$

The starred equality comes form the fact that any real commutes with any quaternion. Finally, take square roots of both sides of the equation. ♠

A quaternion q is called a *unit quaternion* of $|q| = 1$.

Lemma 1.3 *The unit quaternions form a group, with the group law being multiplication.*

Proof: Let G denote the set of unit quaternions. If $q, r \in G$ then $|qr| = |q||r| = 1$ by the previous result. So, G is closed under the operation. The associative law holds because the quaternions form a ring (and also can be checked directly.) The identity element is 1, and the inverse of q is $\overline{q}$. ♠

The group of unit quaternions is often denoted $SU(2)$. The following matrices perfectly mimic the behavior of $1, i, j, k$ and generate a group of matrices isomorphic to the unit quaternion group.

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{4}$$

These matrices are known as *special unitary* 2×2 matrices.

2 Pure Quaternions

The quaternion q in Equation 1 is *pure* if $a = 0$. That is, $q = bi + cj + dk$. We identify the pure quaternions with $\boldsymbol{R}^3$ in the obvious way:

$$bi + cj + dk \leftrightarrow (b, c, d). \tag{5}$$

Let P denote the set of pure quaternions.

Lemma 2.1 *If $q \in SU(2)$ and $r \in P$, then $qrq^{-1} \in P$.*

Proof: $SU(2)$ is generated by the elements $a + bi$, $a + bj$ and $a + bk$. For this reason, it suffices to prove our result when q has one of these forms. By symmetry, it suffices to consider the case when $q = a + bi$. In this case, we want to show that

$$(a + bi)(xi + yj + zk)(a - bi)$$

has no real component. Multiplying out the above expression, we get 12 terms, two of which are real. The two real terms are abx and $-abx$, and they cancel. ♠

By the above result, each $q \in SU(2)$ gives rise to a mapping $T_q : P \to P$. The formula is

$$T_q(p) = qpq^{-1}. \tag{6}$$

Since we have identified P with $\boldsymbol{R}^3$, we think of P as being equipped with the usual notion of Euclidean distance. In terms of our identification, we have

$$\text{distance}(r_1, r_2) = |r_1 - r_2|. \tag{7}$$

An *isometry* is a distance-preserving map.

Lemma 2.2 *T_q is an isometry of P which fixes the origin.*

Proof: First, $T_q(0) = q0q^{-1} = 0$. Second

$$\text{dist}(T_q(r_1), T_q(r_2)) = |qr_1q^{-1} - qr_2q^{-1}| =_*$$

$$|q(r_1 - r_2)q^{-1}| = |q||r_1 - r_2||q^{-1}| = |r_1 - r_2|.$$

The starred equality is the distributive law. So, T_q is an isometry of P which fixes the origin. ♠

3 The Homomorphism

The group $SO(3)$ is the group of orientation preserving isometries of $\boldsymbol{R}^3$ which fix the origin. Since we have identified P with $\boldsymbol{R}^3$, we can equally well think of $SO(3)$ as the group of orientation preserving isometries of P which fix the origin.

Lemma 3.1 *$T_q \in SO(3)$.*

Proof: The only thing left to prove that T_q is orientation preserving. Note that T_q is either orientation preserving or orientation reversing. Note also that, for $q_1, q_2 \in SU(2)$, the corresponding maps T_1 and T_2 are either both orientation preserving or both orientation reversing. Why? Because we can take a continuous path from q_1 to q_2 and the corresponding maps cannot suddenly switch from the one kind to the other.

Finally the map corresponding to $1 \in SU(2)$ is the identity map, and therefore orientation preserving. Hence T_q is orientation preserving for all $q \in SU(2)$. ♠

Define $\Psi : SU(2) \to SO(3)$ by the rule

$$\Psi(q) = T_q. \tag{8}$$

Lemma 3.2 *Ψ is a homomorphism.*

Proof:

$$T_{qr}(p) = (qr)p(qr)^{-1} = q(rpr^{-1})q^{-1} = T_q(T_r(p)).$$

Since this works for all $p \in P$, we have

$$\Psi(qr) = T_{qr} = T_q \circ T_r = \Psi(q)\Psi(r).$$

This does it. ♠

Lemma 3.3 *Ψ is surjective.*

Proof: Let H denote the subgroup of $SU(2)$ consisting of elements of the form $a + bi$. The image $\Psi(H)$ is a subgroup of $SO(3)$ which fixes the pure quaternion ri. That is, $\Psi(H)$ consists of rotations which fix the x-axis. As the coefficients a and b vary, with $a = \cos(\theta)$ and $b = \sin(\theta)$, we produce all such rotations. In short $\Psi(SU(2))$ contains all rotations which fix the x-axis.

By symmetry, $\Psi(SU(2))$ also contains all the rotations which fix the y-axis, and all the rotations which fix the z-axis. But $SO(3)$ is clearly generated by all these rotations. ♠

Lemma 3.4 *Ψ is 2-to-1.*

Proof: Since Ψ is a homomorphism, it suffices to prove that kernel(Ψ) has order 1. The elements 1 and -1 are certainly in the kernel, so we just have to see that these are the only elements in the kernel. Suppose that q lies in the kernel. Then T_q is the identity map. This means that $qp = pq$ for all pure quaternions p.

In particular $qi = iq$. Letting q be as in Equation 1, we compute

$$qi = ai - b - ck + dj, \qquad iq = ai - b + ck - dj.$$

Since these are equal, we must have $c = d = 0$. Similarly, $qj = jq$ forces $b = 0$. Hence $q \in \boldsymbol{R}$. But $SU(2) \cap \boldsymbol{R} = \{1, -1\}$. Hence $q = \pm 1$. ♠

Even though we are done with the proof, there is a bit more to say. The groups $SU(2)$ and $SO(3)$ are also manifolds. Or, if you prefer, they are metric spaces. The map Ψ is continuous with respect to the relevant metrics. This is pretty easy to prove. The point is that the formulas for T_q, considered as a matrix, depend continuously on q.

If you know some topology, you'll appreciate the statement that Ψ is also a covering map. That's why Ψ is called the *spin cover*. The word *spin* comes from particle physics. The additional bit of information recorded by a quaternion is called its spin.