Notes on Tensor Products

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May 3, 2014

1 Modules

Basic Definition: Let R be a commutative ring with 1. A (*unital*) *R-module* is an abelian group M together with a operation $R \times M \to M$, usually just written as rv when $r \in R$ and $v \in M$. This operation is called *scaling*. The scaling operation satisfies the following conditions.

1. $1v = v$ for all $v \in M$.
2. $(rs)v = r(sv)$ for all $r, s \in R$ and all $v \in M$.
3. $(r + s)v = rv + sv$ for all $r, s \in R$ and all $v \in M$.
4. $r(v + w) = rv + rw$ for all $r \in R$ and $v, w \in M$.

Technically, an R-module just satisfies properties $2, 3, 4$. However, without the first property, the module is pretty pathological. So, we'll always work with unital modules and just call them modules. When R is understood, we'll just say *module* when we mean *unital R-module*.

Submodules and Quotient Modules: A *submodule* $N \subset M$ is an abelian group which is closed under the scaling operation. So, $rv \in N$ provided that $v \in N$. A submodule of a module is very much like an ideal of a ring. One defines M/N to be the set of (additive) cosets of N in M, and one has the scaling operation $r(v + N) = (rv) + N$. This makes M/N into another R-module.

Examples: Here are some examples of R-modules.

- When R is a field, an R-module is just a vector space over R.
- The direct product $M_1 \times M_2$ is a module. The addition operation is done coordinate-wise, and the scaling operation is given by

  $$r(v_1, v_2) = (rv_1, rv_2).$$

  More generally, $M_1 \times ... \times M_n$ is another R-module when $M_1, ..., M_n$ are.
- If M is a module, so is the set of finite formal linear combinations $L(M)$ of elements of M. A typical element of $L(M)$ is

  $$r_1(v_1) + ... + r_1(v_n), \qquad r_1, ..., r_n \in R, \quad v_1, ..., v_n \in M.$$

  This definition is subtle. The operations in M allow you to simplify these expressions, but in $L(M)$ you are not allowed to simplify. Thus, for instance, $r(v)$ and $1(rv)$ are considered distinct elements if $r \neq 1$.
- If $S \subset M$ is some subset, then $R(S)$ is the set of all finite linear combinations of elements of S, where simplification is allowed. With this definition, $R(S)$ is a submodule of M. In fact, $R(S)$ is the smallest submodule that contains S. Any other submodule containing S also contains $R(S)$. As with vector spaces, $R(S)$ is called the *span* of S.

2 The Tensor Product

The tensor product of two R-modules is built out of the examples given above. Let M and N be two R-modules. Here is the formula for $M \otimes N$:

$$M \otimes N = Y/Y(S), \qquad Y = L(M \times N), \tag{1}$$

and S is the set of all formal sums of the following type:

1. $(rv, w) - r(v, w)$.
2. $(w, rv) - r(v, w)$.
3. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$.
4. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$.

Our convention is that (v, w) stands for $1(v, w)$, which really is an element of $L(M \times N)$. Being the quotient of an R-module by a submodule, $M \otimes N$ is another R-module. It is called the *tensor product* of M and N.

There is a map $B : M \times N \to M \otimes N$ given by the formula

$$B(m, n) = [(m, n)] = (m, n) + Y(S), \tag{2}$$

namely, the $Y(S)$-coset of (m, n). The traditional notation is to write

$$m \otimes n = B(m, n). \tag{3}$$

The operation $m \otimes n$ is called the *tensor product of elements*.

Given the nature of the set S in the definition of the tensor product, we have the following rules:

1. $(rv) \otimes w = r(v \otimes w)$.
2. $r \otimes (rw) = r(v \otimes w)$.
3. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
4. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$.

These equations make sense because $M \otimes N$ is another R-module. They can be summarised by saying that the map B is *bilinear*. We will elaborate below.

An Example: Sometimes it is possible to figure out $M \otimes N$ just from using the rules above. Here is a classic example. Let $R = \boldsymbol{Z}$, the integers. Any finite abelian group is a module over $\boldsymbol{Z}$. The scaling rule is just $mg = g + ... + g$ (m times). In particular, this is true for $\boldsymbol{Z}/n$. Let's show that $\boldsymbol{Z}/2 \otimes \boldsymbol{Z}/3$ is the trivial module.

Consider the element $1 \otimes 1$. We have

$$2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0(1 \otimes 1) = 0.$$

At the same time

$$2(1 \otimes 1) = 1 \otimes 3 = 1 \otimes 0 = 0(1 \otimes 1) = 0.$$

But then

$$1(1 \otimes 1) = (3 - 2)(1 \otimes 1) = 0 - 0 = 0.$$

Hence $1 \otimes 1$ is trivial. From here it is easy to see that $a \otimes b$ is trivial for all $a \in \boldsymbol{Z}/2$ and $b \in \boldsymbol{Z}/3$. There really aren't many choices. But $\boldsymbol{Z}/2 \otimes \boldsymbol{Z}/3$ is the span of the image of $M \times N$ under the tensor map. Hence $\boldsymbol{Z}/2 \otimes \boldsymbol{Z}/3$ is trivial.

3 The Universal Property

Linear and Bilinear Maps: Let M and N be R-modules. A map $\phi : M \to N$ is *R-linear* (or just *linear* for short) provided that

1. $\phi(rv) = r\phi(v)$.
2. $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$.

A map $\phi : M \times N \to P$ is *R-bilinear* if

1. For any $m \in M$, the map $n \to \phi(m, n)$ is a linear map from N to P.
2. For any $n \in N$, the map $m \to \phi(m, n)$ is a linear map from M to P.

Existence of the Universal Property: The tensor product has what is called a *universal property*. the name comes from the fact that the construction to follow works for all maps of the given type.

Lemma 3.1 *Suppose that $\phi : M \times N \to P$ is a bilinear map. Then there is a linear map $\widehat{\phi} : M \otimes N \to P$ such that $\phi(m, n) = \widehat{\phi}(m \otimes n)$. Equivalently, $\phi = \widehat{\phi} \circ B$, where $B : M \times N \to M \otimes N$ is as above.*

Proof: First of all, there is a linear map $\psi : Y(M \times N) \to P$. The map is given by

$$\psi(r_1(v_1, w_1) + ... + r_n(v_n, w_n)) = r_1\psi(v_1, w_1) + ... + r_n\psi(v_n, w_n). \tag{4}$$

That is, we do the obvious map, and then simplify the sum in P. Since ϕ is bilinear, we see that $\psi(s) = 0$ for all $s \in S$. Therefore, $\psi = 0$ on $Y(S)$. But then ψ gives rise to a map from $M \otimes N = Y/Y(S)$ into P, just using the formula

$$\widehat{\phi}(a + Y(S)) = \psi(a). \tag{5}$$

Since ψ vanishes on $Y(S)$, this definition is the same no matter what coset representative is chosen. By construction $\widehat{\phi}$ is linear and satisfies $\widehat{\phi}(m \otimes n) = \phi(m, n)$. ♠

Uniqueness of the Universal Property: Not only does $(B, M \otimes N)$ have the universal property, but any other pair $(B', (M \otimes N)')$ with the same property is essentially identical to $(B, M \otimes N)$. The next result says this precisely.

Lemma 3.2 *Suppose that $(B', (M \otimes N)')$ is a pair satisfying the following axioms:*

- *$(M \otimes N)'$ is an R-module.*
- *$B' : M \times N \to (M \otimes N)'$ is a bilinear map.*
- *$(M \otimes N)'$ is spanned by the image $B'(M \times N)$.*
- *For any bilinear map $T : M \times N \to P$ there is a linear map $L : (M \otimes N)' \to P$ such that $T = L \circ B'$.*

Then there is an isomorphism $I : M \otimes N \to (M \otimes N)'$ and $B' = I \circ B$.

Proof: Since $(B, M \otimes N)$ has the universal property, and we know that $B' : M \times N \to (M \otimes N)'$ is a bilinear map, there is a linear map $I : M \otimes N \to (M \otimes N)'$ such that

$$B' = I \circ B.$$

We just have to show that I is an isomorphism. Reversing the roles of the two pairs, we also have a linear map $J : (M \otimes N)' \to M \otimes N$ such that

$$B = J \circ B'.$$

Combining these equations, we see that

$$B = J \circ I \circ B.$$

But then $J \circ I$ is the identity on the set $B(M \times N)$. But this set spans $M \otimes N$. Hence $J \circ I$ is the identity on $M \otimes N$. The same argument shows that $I \circ J$ is the identity on $(M \otimes N)'$. But this situation is only possible if both I and J are isomorphisms. ♠

4 Vector Spaces

The tensor product of two vectors spaces is much more concrete. We will change notation so that F is a field and V, W are vector spaces over F. Just to make the exposition clean, we will assume that V and W are finite dimensional vector spaces. Let $v_1, ..., v_m$ be a basis for V and let $w_1, ..., w_n$ be a basis for W. We define $V \otimes W$ to be the set of formal linear combinations of the mn symbols $v_i \otimes w_j$. That is, a typical element of $V \otimes W$ is

$$\sum_{i,j} c_{ij}(v_i \otimes w_j). \tag{6}$$

The space $V \otimes W$ is clearly a finite dimensional vector space of dimension mn. it is important to note that we are not giving a circular definition. This time $v_i \otimes w_j$ is just a formal symbol.

However, now we would like to define the bilinear map

$$B : V \times W \to V \otimes W.$$

Here is the formula

$$B\Big(\sum a_i v_i, \sum b_j w_j\Big) = \sum_{i,j} a_i b_j (v_i \otimes w_j). \tag{7}$$

This gives a complete definition because every element of V is a unique linear combination of the $\{v_i\}$ and every element of W is a unique linear combination of the $\{w_j\}$. A routine check shows that B is a bilinear map.

Finally, if $T : V \times W \to P$ is some bilinear map, we define $L : V \otimes W \to P$ using the formula

$$L\Big(\sum_{i,j} c_{ij}(v_i \otimes w_j)\Big) = \sum_{i,j} c_{ij} T(v_i, w_j). \tag{8}$$

It is an easy matter to check that L is linear and that $T = L \circ B$.

Since our definition here of B and $V \otimes W$ satisfies the universal property, it must coincide with the more abstract definition given above.

5 Properties of the Tensor Product

Going back to the general case, here I'll work out some properties of the tensor product. As usual, all modules are unital R-modules over the ring R.

Lemma 5.1 *$M \otimes N$ is isomorphic to $N \otimes M$.*

Proof: This is obvious from the construction. The map $(v, w) \to (w, v)$ extends to give an isomorphism from $Y_{M,N} = L(M \times N)$ to $Y_{N,M} = L(N \times N)$, and this isomorphism maps the set $S_{M,N} \subset Y_{M,N}$ of bilinear relations set $S_{N,M} \subset Y_{N,M}$ and therefore gives an isomorphism between the ideals $Y_{M,N}S_{M,N}$ and $Y_{N,M}S_{N,M}$. So, the obvious map induces an isomorphism on the quotients. ♠

Lemma 5.2 *$R \otimes M$ is isomorphic to M.*

Proof: The module axioms give us a surjective bilinear map $T : R \times M \to M$ given by $T(r, m) = rm$. By the universal property, there is a linear map $L : R \otimes M \to M$ such that $T = L \circ B$. Since T is surjective, L is also surjective. At the same time, we have a map $L^* : M \to R \otimes M$ given by the formula

$$L^*(v) = B(1, v) = 1 \otimes v. \tag{9}$$

The map L^* is linear because B is bilinear. We compute

$$L^* \circ L(r \otimes v) = L^*(rv) = 1 \otimes rv = r \otimes v. \tag{10}$$

So $L^* \circ L$ is the identity on the image $B(R \times M)$. But this image spans $R \otimes M$. Hence $L^* \circ L$ is the identity. But this is only possible if L is injective. Hence L is an isomorphism. ♠

Lemma 5.3 *$M \otimes (N_1 \times N_2)$ is isomorphic to $(M \otimes N_1) \times (N \otimes N_2)$.*

Proof: Let $N = N_1 \times N_2$. There is an obvious isomorphism ϕ from $Y = Y_{M,N}$ to $Y_1 \times Y_2$, where $Y_j = Y_{M,N_j}$, and $\phi(S) = S_1 \times S_2$. Here $S_j = S_{M,N_j}$. Therefore, ϕ induces an isomorphism from Y/YS to $(Y_1/Y_1S_1) \times (Y_2/Y_2S_2)$. ♠

Finally, we can prove something (slghtly) nontrivial.

Lemma 5.4 *$M \otimes R^n$ is isomorphic to M^n.*

Proof: By repeated applications of the previous result, $M \otimes R^n$ is isomorphic to $(M \otimes R)^n$, which is in turn isomorphic to M^n. ♠

As a special case,

Corollary 5.5 *$R^m \otimes R^n$ is isomorphic to R^{mn}.*

This is a reassurance that we got things right for vector spaces.
For our next result we need a technical lemma.

Lemma 5.6 *Suppose that Y is a module and $Y' \subset Y$ and $I \subset Y$ are both submodules. Let $I' = I \cap Y'$. Then there is an injective linear map from Y'/I' into Y/I.*

Proof: We have a linear map $\phi : Y' \to Y/I$ induced by the inclusion from Y' into Y. Suppose that $\phi(a) = 0$. Then $a \in I$. But, at the same time $a \in Y'$. Hence $a \in I'$. Conversely, if $a \in I'$ then $\phi(a) = 0$. In short, the kernel of ϕ is I'. But then the usual isomorphism theorem shows that ϕ induces an injective linear map from Y'/I' into Y/I. ♠

Now we deduce the corollary we care about.

Lemma 5.7 *Suppose that $M' \subset M$ and $N' \subset N$ are submodules. Then there is an injective linear map from $M' \otimes N'$ into $M \otimes N$. This map is the identity on elements of the form $a \otimes b$, where $a \in M'$ and $b \in N'$.*

Proof: We apply the previous result to the module $Y = Y_{M,N}$ and the submodules $I = S_{M,N}$ and $M' = Y_{M',N'}$. ♠

In view of the previous result, we can think of $M' \otimes N'$ as a submodule of $M \otimes N$ when $M' \subset N$ and $N' \subset N$ are submodules.

This last result says something about vector spaces. Let's take an example where the field is $\boldsymbol{Q}$ and the vector spaces are $\boldsymbol{R}$ and $\boldsymbol{R}/\boldsymbol{Q}$. These two vector spaces are infinite dimensional. It follows from Zorn's lemma that they both have bases. However, You might want to see that $\boldsymbol{R} \otimes \boldsymbol{R}/\boldsymbol{Q}$ is nontrivial even without using a basis for both. If we take any finite dimensional subspaces $V \subset \boldsymbol{R}$ and $W \subset \boldsymbol{R}/\boldsymbol{Q}$, then we know $V \otimes W$ is a submodule of $\boldsymbol{R} \otimes \boldsymbol{R}/\boldsymbol{Q}$. Hence $\boldsymbol{R} \otimes \boldsymbol{R}/\boldsymbol{Q}$ is nontrivial. In particular, we can use this to show that the element $1 \otimes [\alpha]$ is nontrivial when α is irrational.