The purpose of this homework assignment is to give some additional information about the objects related to Weierstrass uniformization. If you read the last set of notes, you will find some answers, or at least partial answers, to some of these problems.

Let $SL_2(\boldsymbol{Z})$ denote the group of 2×2 integer matrices with determinant 1. This group is known as the *modular group*. Let $\boldsymbol{H}^2$ denote the set of complex numbers of the form $x + iy$ where $y > 0$. So, $\boldsymbol{H}^2$ is the open upper half plane. $\boldsymbol{H}^2$ is also known as the *hyperbolic plane*.

There is a natural *group action* of $SL_2(\boldsymbol{Z})$ on $\boldsymbol{H}^2$. Given

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\boldsymbol{Z}),$$

we define

$$T_M(z) = \frac{az + b}{cz + d}.$$

Exercise 1: Check on a non-trivial example that $T_A(T_B(z)) = T_{AB}(z)$, where AB denotes matrix multiplication. You could also prove it in general, but this is messy.

As is usual, we can abuse notation slightly and write $M(z) = T_M(z)$, when $M \in SL_2(\boldsymbol{Z})$. That is, we think of $SL_2(\boldsymbol{Z})$ as "acting directly" on $\boldsymbol{H}^2$. We define the *orbit* if a point $z \in \boldsymbol{H}^2$ to be the set $\{g(z) |\ g \in SL_2(\boldsymbol{Z})\}$.

Recall that a *lattice* is a set of the form

$$\Lambda(\alpha, \beta) = \{m\alpha + n\beta |\ n, m \in \boldsymbol{Z}\}$$

with $\alpha/\beta \notin \boldsymbol{R}$. Note that either $\alpha/\beta \in \boldsymbol{H}^2$ or $\beta/\alpha \in \boldsymbol{H}^2$. By convention, we order α and β so that $\beta/\alpha \in \boldsymbol{H}^2$. Say that two lattices Λ_1 and Λ_2 are *equivalent* if and only if there is some complex number $w \neq 0$ so that $w\Lambda_1 = \Lambda_2$. In other words, the one grid of points is just a scaled/rotated copy of the other.

Exercise 2. Prove that every lattice is equivalent to a lattice of the form $\Lambda(1, z)$, with $z \in \boldsymbol{H}^2$, and that $\Lambda(1, z)$ and $\Lambda(1, z')$ are equivalent if and only if they lie in the same $SL_2(\boldsymbol{Z})$ orbit. You can find the basic ideas for the proof in the notes, but try to do it without looking it up.

According to Exercise 2, the "space of equivalence classes" of lattices is the same as the space of $SL_2(\boldsymbol{Z})$ orbits in $\boldsymbol{H}^2$. This space is typically written as $\boldsymbol{H}^2/SL_2(\boldsymbol{Z})$. In light of the Weierstrass uniformization result, this space is often called *the moduli space of elliptic curves*. We're going to picture it.

Exercise 3: Say that two fractions p_1/q_1 and p_2/q_2 are *Farey related* if

$$\pm\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \in SL_2(\boldsymbol{Z}).$$

I mean that one sign choice or the other works. For instance $1/2$ and $2/3$ are Farey related. Define

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1+p_2}{q_1+q_2}.$$

This operation is known as *Farey addition*. If you added fractions this way in fourth grade, you were sent to the corner of the room, but amazingly this operation is closely related to the modular group and the space of lattices. Prove the following statement. If r_1 and r_2 are two Farey related fractions, then $r_1 \oplus r_2$ is Farey related to both r_1 and r_2.

Exercise 4: The *Farey graph* is a graph whose vertices are the elements of $\boldsymbol{Q} \cup \infty$. (It is nice to think of $\boldsymbol{Q} \cup \infty$ as the projective line over $\boldsymbol{Q}$.) The point ∞ is interpreted as the fraction $1/0$, and the integer n is interpreted as $n/1$. Two vertices are joined by an edge if and only if they are Farey related. For instance ∞ is joined to every integer. You can get a nice picture of the Farey graph by representing the edge connecting $1/0$ to $n/1$ as a vertical ray starting at the point $(n,0)$ and representing the remaining edges as semicircles with endpoints on the real line. Draw enough of the Farey graph so that you see what it looks like. Hint: use Exercise 3 to generate lots of edges.

Exercise 5: Call a curve in $\boldsymbol{H}^2$ a *geodesic* if it is either an open semicircle with endpoints on the real line, or else a vertical ray. So, the edges of the Farey graph are made from geodesics. Let's take it as an axiom that $SL_2(\boldsymbol{Z})$ acts on $\boldsymbol{H}^2$ in such a way that it maps geodesics to geodesics. (One can prove this by direct calculation.) Assuming this axiom, prove that $SL_2(\boldsymbol{Z})$ acts on $\boldsymbol{H}^2$ so as to induce a group of isomorphisms of the Farey graph. All you have to prove here is the following: If r_1 and r_2 are Farey related fractions, then so are $T_M(r_1)$ and $T_M(r_2)$ for any $M \in SL_2(\boldsymbol{Z})$.

It is worth mentioning that there is a special way of measuring distances on $\boldsymbol{H}^2$, known as the *Hyperbolic metric*, so that the geodesics become the 'straight lines". With this interpretation, the Farey graph is a hyperbolic geometry version of a regular tesselation and $SL_2(\boldsymbol{Z})$ is its symmetry group.

Exercse 6: Say that a *Farey triangle* is a triple of edges that cyclically share an endpoint. For instance, the edges joining 0, 1, and ∞ make a Farey triangle. Look carefully at the Farey graph and notice that all of $\boldsymbol{H}^2$ is filled with Farey triangles. Color some of these black and white in an alternating pattern so as to make a pretty picture.

Exercise 7: By construction, $SL_2(\boldsymbol{Z})$ acts on $\boldsymbol{H}^2$ so as to permute the Farey triangles. Let τ_0 be the Farey triangle from Exercise 6. Let τ be any other Farey triangle. Prove that there is some $M \in SL_2(\boldsymbol{Z})$ such that $T_M(\tau) = \tau_0$. Hint: prove this when τ and τ_0 share an edge, and then use induction on the "combinatorial distance" between τ and τ_0, in general.

Exercise 8: Let $\Gamma_3 \subset SL_2(\boldsymbol{Z})$ denote the subgroup consisting of elements M such that $T_M(\tau_0) = \tau_0$. Prove that Γ_3 has order 3. The elements of Γ_3 cyclically permute the vertices of τ_0 and act as "rotations".

Exercise 9: Let E be the edge of the Farey graph that joins 0 to ∞. Let $\Gamma_2 \subset SL_2(\boldsymbol{Z})$ denote those elements M such that $T_M(E) = E$. Prove that Γ_2 has order 2. The nontrivial element reverses the endpoints of E.

Exercise 10: Put everything together and argue that $\boldsymbol{H}^2/SL_2(\boldsymbol{Z})$ is homeomorphic to the space with the following description: Start with a solid equilateral triangle T and remove the vertex. Let G_3 denote the group of rotations of T. Define $p, q \in T$ to be equivalent if p, q are in the same G_3 orbit. If $p, q \in \partial T$ (the boundary) say additionally that $p \sim q$ if p and q lie on the same edge, at the same distance from the midpoint. Convince yourself (and the grader) that $T/\sim$ is homeomorphic to a plane. Hint: Divide T into 3 nice pieces. Let T' be one of the pieces. Every equivalence class intersects T'. In the interior of T', each equivalence class is uniquely represented. On the boundary of T', each equivalence class has 2 representatives, and these must be "sewn together". When you do the sewing, T' folds up like a taco. You would get a sphere, but the vertex is removed. So, you get a plane.