Math 154 Notes 2

In these notes I'll construct some explicit transcendental numbers. These numbers are examples of Liouville numbers.

Let $\{a_k\}$ be a sequence of positive integers, and suppose that a_k divides a_{k+1} for all k. We say that $\{a_k\}$ has *moderate growth* if there is some constant K such that $a_{n+1} < K(a_n)^K$ for all n. Otherwise we say that $\{a_n\}$ has *immoderate growth.*

Theorem 0.1 *If $\{a_k\}$ has immoderate growth then the number $x = \sum_{n=1}^{\infty} \frac{1}{a_n}$ is transcendental.*

Proof: We will suppose that $P(x) = 0$ for some polynomial P. There is some constant C_1 such that P has degree at most C_1 and all the coefficients of P are less than C_1 in absolute value. Let $x_n = 1/a_1 + ... + 1/a_n$. The divisibility condition guarantees that $x_n = h_n/a_n$, where h_n is some integer. As long as $P(x_n)$ is nonzero, we have

$$|P(x_n) - P(x)| = |P(x_n)| \geq (a_n)^{-C_1}. \tag{1}$$

This just comes from the fact that $P(x_n)$ is a rational number whose denominator is at most $(a_n)^{C_1}$.

On the other hand, there is some constant C_2 such that $|P'(u)| < C_2$ for all u in the interval $[x-1, x+1]$. For n large, x_n lies in this interval. Therefore, by the usual estimate that comes from integration,

$$|P(x_n) - P(x)| < C_2|x_n - x| < 2C_2 a_{n+1}^{-1}. \tag{2}$$

The last inequality comes from the fact that $\{1/a_n\}$ decays faster than the sequence $\{1/2^n\}$ once n is large.

Combining Equations 1 and 1 gives

$$(a_n)^{-C_1} \leq 2C_2 a_{n+1}^{-1}.$$

This hoolds for all sufficiently large n. From here its easy to see that $\{a_n\}$ has moderate growth. This is a contradiction. ♠

As I said in class, an explicit example is given by $a_n = 10^{n!}$. Hence the number 0.0100100000010000000000000000000001... is transcendental.

There are a few more things I want to say about this construction. In order to get Equation 1, I had to use the divisibility condition. The key point was that $x_n = h_n/a_n$. In class I was a bit muddled on this point, and I want to clear it up.

Suppose that we don't have the divisibility condition. Let's just assume the mild condition that the numbers $\{a_n\}$ are increasing. Then, at worst,

$$x_n = \frac{h_n}{\alpha_n}; \qquad \alpha_n < (a_n)^n. \tag{3}$$

This gives the weaker inequality

$$|P(x) - P(x_n)| \leq (a_n)^{-nC_1}. \tag{4}$$

Equations 4 and 1 lead to the condition that

$$a_{n+1} < K a_n^{nK}, \tag{5}$$

which is not as strong.

We can still construct transcendental numbers by taking the sequence $\{a_n\}$ to grow so fast that Equation 5 does not hold for any constant K. An example would be

$$a_1 = 1; \qquad a_{n+1} = 10^{a_n}.$$

This is an insanely fast-growing sequence. It will hurt your mind to think about the number $x = \sum 1/a_n$ in this case, but it certainly satisfies the criterion.

Even though I've given examples of Liouville numbers, I haven't defined them exactly. The quickest definition is that x is a Liouville number if, for every n, there is a rational number p/q such that $|x - p/q| < q^{-n}$. This is perhaps an easier condition to state than the special case I mentioned above, but maybe it is not as easy to verify the condition directly. (Or maybe I should have just done it this way in class!) Anyway, try to prove directly that any Liouville number is transcendental.