The purpose of these notes is to prove a special case of the Cayley-Bacharach Theorem and then to prove Pascal's Theorem as an application. The main result we prove, the Grid Theorem, will be useful when we analyze the group structure of an elliptic curve.

1 A Preliminary Result about Conics

Let $\boldsymbol{F}$ be a field. Say that 4 points in $P^2(\boldsymbol{F})$ are *in general position* if no 3 of those points are collinear. Recall that a *conic* in $P^2(\boldsymbol{F})$ is the solution to a homogeneous polynomial of degree 2.

Lemma 1.1 *Let A_1, A_2, A_3, A_4 be 4 general position points and let B be some 5th point. There exists a conic that contains A_j for all j but not B.*

Proof: Let L_1 be the line containing A_1 and A_2 and let L_2 be the line containing A_3 and A_4. Suppose first that B lies on neither L_1 nor L_2. There is a homogeneous degree 1 polynomial λ_j such that L_j is the projective curve corresponding to λ_j. That is, $L_j = V_{\lambda_j}$. Let $\lambda = \lambda_1\lambda_2$. This is a homogeneous degree 2 polynomial that vanishes exactly on $L_1 \cup L_2$, and hence not on B.

Suppose that $B \in L_1$. This time we let L_1' be the line containing A_1 and A_3 and L_2' be the line containing A_2 and A_3. Suppose $B \in L_1'$. Then L_1' contains both A_2 and B. But L_1 contains both A_2 and B. Hence $L_1 = L_1'$. Hence A_1, A_2, A_3 are collinear. This contradiction shows that $B \notin L_1'$. A similar argument shows that $B \notin L_2'$. Now we can repeat the original argument, using L_1' and L_2' in place of L_1 and L_2. ♠

2 The Grid Theorem

The results in this section work for any field.

Theorem 2.1 *Suppose that a homogeneous curve of degree at most 3 contains 8 points of a grid. Then it also contains the 9th point.*

Here is an equivalent formulation. Say that a *vector grid* is a collection of 9 vectors in $\boldsymbol{F}^3$ representing the points of a grid in $P^2(\boldsymbol{F})$.

Theorem 2.2 *A homogeneous polynomial of degree at most* 3 *that vanishes on* 8 *vectors of a vector grid also vanishes on the* 9*th vector.*

Let V denote the set of homogeneous curves of degree 3. As a vector space V is isomorphic to $\boldsymbol{F}^{10}$. To see this, note that an element of V is specified by choosing constants $a_1, ..., a_{10} \in \boldsymbol{F}$, which give rise to the polynomial $a_1X^3 + a_3Y^3 + ... + a_{10}XYZ$. Given any nonzero vector $v \in \boldsymbol{F}^3$, let $S_v \subset V$ denote those homogeneous polynomials that vanish on v. Note that S_v is a linear subspace of V. Below, I'll prove the following result.

Lemma 2.3 *Let $v_1, ..., v_8$ be* 8 *vectors of the grid. Let $S_j = S_{v_j}$. Then the intersection $S_1 \cap ... \cap S_8$ is* 2 *dimensional.*

Proof: To each subspace S_j we have a vector V_j such that S_j is the solution of the equation $(a_1, ..., a_{10}) \cdot V_j = 0$. Lemma 2.3 is equivalent to the statement that the vectors $V_1, ..., V_8$ are linearly independent. We will suppose this is not the case, and derive a contradiction.

If our vectors $V_1, ..., V_8$ are not independent, then (after relabelling) we can write V_8 as a linear combination of $V_1, ..., V_7$. This is the same thing as saying that $S_1 \cap ... \cap S_7 \subset S_8$. In other words, any homogeneous polynomial of degree at most 3 that vanishes on $v_1, ..., v_7$ also vanishes on v_8. We will get a contradiction by producing a homogeneous polynomial of degree 3 that vanishes on $v_1, ..., v_7$ but not on v_8.

Here is the key observation. A case by case analysis shows that we can divide up the points $[v_1], ..., [v_7]$ so that (after relabelling if necessary)

- $[v_1], [v_2], [v_3]$ all lie on the line L. Here L is one of the 6 special lines defining the grid. Note that $[v_8]$ does not lie on L.
- $[v_4], [v_5], [v_6], [v_7]$ are in general position: No 3 are collinear.

Since $\boldsymbol{F}$ is a nice field, we can then find a conic section M that contains these 4 points but does not contain the point $[v_8]$.

The line L is the projective curve associated to a homogeneous polynomial λ of degree 1. The ellipse M is the projective curve associated to a homogeneous polynomial of degree 2. The homogeneous cubic $\lambda\mu$ vanishes on $[v_1], ..., [v_7]$ but not on $[v_8]$. ♠

Each line A_i is the projective curve corresponding to a degree 1 homogeneous polynomial α_i. Likewise, each line B_i is the projective curve corresponding to a degree 1 homogeneous polynomial β_i. Let $\alpha = \alpha_1\alpha_2\alpha_3$ and $\beta = \beta_1\beta_2\beta_3$. Note that α and β are both homogeneous cubics which vanish on all 9 grid vectors. Note also that α and β are linearly independent (as elements of V) because α vanishes on $A_1 \cup A_2 \cup A_3$ and β vanishes on $B_1 \cup B_2 \cup B_3$.

Here is the punchline: The set of polynomials of the form

$$\Sigma = \{a\alpha + b\beta; \quad a, b \in \boldsymbol{F}\} \tag{1}$$

is a 2 dimensional set that vanishes on $v_1, ..., v_8$. Hence $S_1 \cap ... \cap S_8 = \Sigma$. But, every element of Σ vanishes on the 9th vector as well. Hence, every element of $S_1 \cap ... \cap S_8$ also vanishes on the 9th vector.

3 Pascal's Theorem

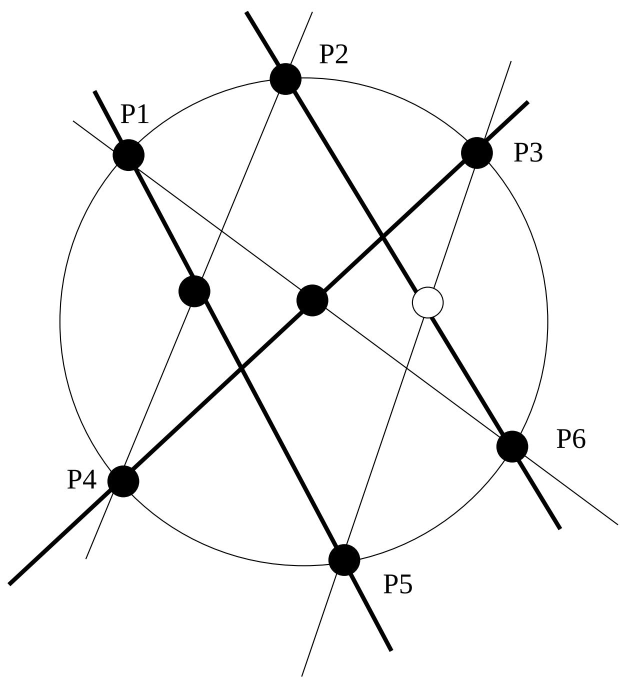

Figure 1: Pascal's Theorem

Pascal's Theorem refers to the configuration in Figure 1. The 6 points $P_1, ..., P_6$ lie on a conic M, and the theorem is that the points X_1, X_2, X_3 lie on a line. Let L be the line containing X_1 and X_2. We want to see that $X_3 \in L$. The line L is the projective curve associated to a homogeneous degree 1 polynomial λ. Likewise, the conic M is the projective curve associated to a homogeneous degree 2 polynomial μ. The polynomial $P = \lambda\mu$ vanishes on $L \cup M$. Hence P vanishes on $P_1, ..., P_6, X_1, X_2$.

The points $P_1, ..., P_6, X_1, X_2, X_3$ make a grid. Hence, by the Grid Theorem, P vanishes on X_3. But P vanishes exactly on $L \cup M$. Hence $X_3 \in L \cup M$. But a line intersects a conic at most twice, by Bezout's Theorem. Hence $X_3 \in L$. This completes the proof.