Math 181 Handout 1

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The purpose of this first handout is to make sense of the following

Definition: A *surface* is a metric space which satisfies the following property. every point on the surface has a neighborhood which is homeomorphic to the plane.

Don't worry if you don't know what some of the words in the above definition mean. The point of this handout is to explain what they mean.

1 A Word about Sets

A *set* is an undefined notion for this class. Informally, a *set* is a collection of things, called *elements*. You should be familiar with such sets as

- $\mathbf{Z}$, the integers.
- $\mathbf{N} = \{1, 2, 3...\}$, the natural numbers.
- $\boldsymbol{R}$, the real numbers.

A *map* between sets A and B is a rule, say f, which assigns to each element $a \in A$, an element $b = f(a) \in B$. This is usually written as $f : A \to B$. The map f is *one to one* if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$. The map f is *onto* if the set $\{f(a)|\ a \in A\}$ equals B. The map f is a *bijection* if it is both one-to-one and onto. Two sets are *bijective* if there is some bijection between them. All the sets we consider will be bijective to either a finite set, or $\mathbf{N}$ or $\boldsymbol{R}$.

The product $A \times B$ of sets is the set of ordered pairs (a, b) with $a \in A$ and $b \in B$. In particular, $\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R}$ is the plane.

2 Metric Spaces

A *metric space* is a set X together with a map $d : X \times X \to \boldsymbol{R}$ such that

- (nondegeneracy) $d(x, y) \geq 0$ for all x, y, with equality iff $x = y$.
- (symmetry) $d(x, y) = d(y, x)$ for all x, y.
- (triangle inequality) $d(x, z) \leq d(x, y) + d(z, y)$ for all x, y, z.

d is called a metric on X. Note that the same set can have many different metrics.

Some examples of metric spaces:

- **Exercise 1:** Let $X = \boldsymbol{R}^2$, the plane. Define the *dot product*
$$V \cdot W = v_1 w_1 + v_2 w_2.$$
Here $V = (v_1, v_2)$ and $W = (w_1, w_2)$. Also define
$$\|V\| = \sqrt{V \cdot V}.$$
Finally, define $d(V, W) = \|V - W\|$. Prove that d is a metric on $\boldsymbol{R}^2$. Hint: You might want to use the *Cauchy-Schwarz inequality*
$$|V \cdot W| \leq \|V\| \|W\|.$$
The metric in this exercise is known as the *Euclidean metric* on $\boldsymbol{R}^2$, or else the *standard metric*.

- **Exercise 2:** On $\mathbf{Z}$ define $d(m, n) = 2^{-k}$ where k is such that 2^k divides $|m - n|$ but 2^{k+1} does not. Also define $d(m, m) = 0$. For instance $d(3, 7) = 2^{-2} = 1/4$ because 2^2 divides 4 but $2^3 = 8$ does not. Prove that d is a metric on $\mathbf{Z}$. This metric is called the 2-adic metric. It is pretty different from the usual metric on the integers.

- If X is a metric space and $Y \subset X$ is a subset, then the metric on X automatically defines a metric on Y, by restriction. For instance, any subset of the plane automatically can be interpreted as a metric space, using the metric from Exercise 1.

- Given any set X define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. This is called the *discrete metric* on X. It is a pretty boring example.

3 Open and Closed Sets

Let X be a metric space with metric d. An *open ball* in X is a subset of the form

$$\{x|\ d(x,c) < r\}.$$

Here c is the *center* of the ball and r is the radius. Say that a subset $U \subset X$ is *open* if for every point $x \in U$ there is some open ball B_x such that $x \in B_x$ and $B_x \subset U$. Note that open balls are open sets.

Exercise 3: Prove that the intersection of two open sets is open. Prove also that the arbitrary union of open sets is open.

Here is some vocabulary, which will be familiar to you if you've had a real analysis class:

- The notation $X - A$ means the complement of A in X, namely the set of points in X which are not in A.

- Given a point $x \in X$ a *neighborhood* of x is any open subset $U \subset X$ such that $x \in X$. For instance, the ball of radius r about x is a perfectly good neighborhood of x.

- The *interior* of a set $A \subset X$ is the union of all open subsets of A. By Exercise 3, the interior of a set is open. Sometimes the interior of A is denoted as A^o. Put another way A^o is the largest open set contained in A.

- A set $C \subset X$ is *closed* if $X - C$ is open.

- The *closure* of a set A is the set

  $$\overline{A} = X - (X - A)^o.$$

  Put another way, $\overline{A}$ is the smallest closed set which contains A.

- The *boundary* of A is the set

  $$\partial A = \overline{A} - A^o.$$

- A set $A \subset X$ is *dense* if $\overline{A} = X$. For instance, the set of rational numbers is dense in the set of real numbers.

4 Continuous Maps

A *map* between metric spaces is just a map in the set theoretic sense. There are two equivalent definitions of continuity for maps between metric spaces. The first one is much cleaner but the second one is probably more familiar.

Definition 1: The map $f : X \to Y$ is *continuous* if it has the following property: For any open $V \subset Y$ the set

$$U = f^{-1}(V) := \{x|\ f(x) \in V\}$$

is an open set of X.

Definition 2: First f is continuous at $x \in X$ if, for any $\epsilon > 0$, there is some $\delta > 0$ such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \epsilon$. Here d_X is the metric on X and d_Y is the metric on Y. Then f is continuous on X if f is continuous at each point of X.

Exercise 4: Show that the two definition of continuity coincide.

Now let X, Y, Z all be metric spaces. Let $f : X \to Y$ be a map and let $g : Y \to Z$ be map. The composition $h = g \circ f$ is defined as $h(x) = g(f(x))$. So h is a map from X to Z.

Lemma 4.1 *The composition of continuous maps is continuous.*

Proof: The first definition is much better for this. Let W be an open subset of Z. Our goal is to show that $h^{-1}(W)$ is open in X. Note that $h^{-1}(W) = f^{-1}(V)$ where $V = g^{-1}(W)$. Since g is continuous, V is open. Since f is continuous and V is open, U is open. This works for any choice of open W, so we are done. ♠

Exercise 5 Give an example of metric spaces X and Y, and $f : X \to Y$ such that

- f is a bijection.
- f is continuous.
- f^{-1} (the inverse map) is not continuous.

This is a classic problem.

5 Homeomorphisms

Let X and Y be two metric spaces. A map $h : X \to Y$ is a *homeomorphism* if

- h is a bijection
- h is continuous.
- h^{-1} is continuous.

Compare Exercise 5. The spaces X and Y are said to be *homeomorphic* if there is some homeomorphism from X to Y. Intuitively, two sets are homeomorphic if one can be "warped" into the other one. Often we don't care exactly which metric we are using, but we just bring in the metric to be able to talk about things like continuity and open sets. Another way to "throw out the metric" is to introduce the notion of a *topological space*. In some ways topological spaces are easier to work with than metric spaces, and more flexible, but they are more abstract. If you're interested in this, let me know.

Even though sets might look very different to the eye, they might be homeomorphic. The next exercise gives some examples of this.

Exercise 6 Prove that the following subsets of the plane (with the standard metric) are all homeomorphic to each other:

- An open ball.
- The interior of a (filled in) triangle.
- The plane itself.

Exercise 7: We can give $\boldsymbol{R}$ the standard metric $d(x, y) = |x - y|$. Prove that $\boldsymbol{R}$ is not homeomorphic to $\boldsymbol{R}^2$, with its standard metric.

Exercise 8 (Challenge) You can imitate the construction in Exercise 1 to put a metric on $\boldsymbol{R}^3$, three dimensional space. Prove that $\boldsymbol{R}^2$ is not homeomorphic to $\boldsymbol{R}^3$. As it turns out $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ are homeomorphic if and only if $m = n$. When you try to prove something like this, you start getting into algebraic topology.

6 Surfaces

Now let's go back to the original definition. Let X be a surface. This means, first of all, that X is a metric space. So, it makes sense to talk about open and closed sets on X and also continuous functions from X to other metric spaces. So, what makes X is surface is that each point $x \in X$ has an open neighborhood U such that U is homeomorphic to $\boldsymbol{R}^2$. You should picture U as a little open disk drawn around x. So X has the property that, around every point, it "looks" like the plane.

Exercise 9: The unit sphere S^2 in $\boldsymbol{R}^3$ is the set $\{(x,y,z)|\ x^2+y^2+z^2=1\}$. This set inherits a metric from $\boldsymbol{R}^3$. Prove that S is a surface, according to our definition. So, for each point $x \in S$ you need to find an open subset $U_x \subset S$ and also a map $f_x : U_x \to \mathbf{R^2}$ which is a homeomorphism. (Hint: Try to use symmetry to reduce the problem to showing that just one point in S^2 has the desired neighborhood.)

Exercise 10: Consider the following subset of $\boldsymbol{R}^4$:

$$T^2 = \{(x,y,z,w)|\ x^2+y^2=1; \quad z^2+w^2=1\}.$$

This set inherits a metric from $\boldsymbol{R}^4$. You might recognize T^2 as the product of two circles. Prove that T^2 is a surface. This surface is known as a *torus*. (Hint: again, try to use symmetry.)

In this class, we'll construct many more examples of surfaces besides the ones in Exercises 9 and 10. In the next handout, I'll explain a general construction, called *gluing* which lets you make lots of surfaces.