Math 181 Handout 10

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The purpose of this handout is to give some basic material on complex analytic functions. All this material can be found in any book on the subject.

1 Basic Definitions

Throughout the handout U will denote an open subset of $\boldsymbol{C}$, the complex plane. Let $f : U \to \boldsymbol{C}$ be a continuous map. We say that f *has a complex derivative* at $z \in U$ if the quotient

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

exists and is finite. Note that h is allowed to be a complex number. f is said to be *complex analytic* in U if $f'(z)$ exists for all $z \in U$ and the function $z \to f'(z)$ varies continuously in U. Complex analytic functions are sometimes called *holomorphic functions*. The two terms are synonyms.

Here are some examples:

- The function $f(z) = z$ is complex analytic in $\boldsymbol{C}$.
- **Exercise 1:** Suppose that f and g are complex analytic in U and g is never 0 in U. Prove that the functions $f+g$ and $f-g$ and fg and f/g are all complex analytic in U.
- Let P and Q be polynomials with complex coefficients. Using Exercise 1 and induction, we see that tha *rational function* $f(z) = P(z)/Q(z)$ is complex analytic in $\boldsymbol{C} - S$, where S is the set of zeros of Q.

- If f is complex analytic on U and g is complex analytic on V and $f(U) \subset V$ then $g \circ f$ is complex analytic and the complex derivative satisfies the chain rule:

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

- **Exercise 2:** Prove that the function $f(z) = z^2 + 3\overline{z}$ is not complex analytic in $\boldsymbol{C}$. So, not all smooth maps are complex analytic .

2 The Cauchy-Riemann Equations

We can think of a complex analytic function f as a map from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$ by writing

$$f(x+iy) = u(x+iy) + iv(x+iy).$$

Recall that f is differentiable at the point (x, y) if the matrix of partial derivatives

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

exists at $p = (x, y)$ and

$$\lim_{t \to 0} \frac{f(p+tv) - f(v)}{t} = df|_p(v).$$

Here $t \in \boldsymbol{R}$. To say that f has a complex derivative at $z = x+iy$ is the same as saying that f is differentiable and $df|_p$ is the composition of a rotation and a dilation. That is

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}; \qquad r \in \boldsymbol{R}, \quad \theta \in [0, 2\pi).$$

Equating terms, we get

$$u_x = v_y; \quad u_v = -v_x.$$

These are called the *Cauchy-Riemann equations*. Thus, if f is complex analytic, then its first partials vary continuously and satisfy the Cauchy-Riemann equations.

The converse is also true: f is complex analytic provided that df exists, is continuous, and saisfies the Cauchy-Riemann equations.

3 Power Series

We say that a sequece $\{a_n\}$ of complex numbers satisfies the *unit convergence condition* (or UCC) if

$$\lim_{n\to\infty} a_n \rho^n = 0$$

for all $\rho \in [0,1)$. The UCC implies that the terms in the sequence $\{|a_n|\rho^n\}$ decay exponentially fast for any $\rho < 1$. To see this, we choose any $\rho^* \in (\rho, 1)$ and note that

$$|a_n|\rho^n = |a_n|(\rho^*)^n \times \left(\frac{\rho}{\rho^*}\right)^n < \left(\frac{\rho}{\rho^*}\right)^n$$

for n sufficiently large.

Exercise 3: Suppose that $\{a_n\}$ satisfies the UCC. Prove that $\{n\delta a_n\}$ also does. Here δ is any contstant you like.

Define

$$f_N(z) = \sum_{n=0}^{N} a_n z^n.$$

If $a, b > N$ and N is sufficiently large then

$$|f_a(z) - f_b(z)| = |\sum_{n=a}^{b} a_n z^n| \leq \sum_{n=a}^{b} |a_n||z|^n \leq \sum_{N}^{\infty} \delta^n = \frac{\delta^n}{1-\delta}.$$

Here we have chosen some $\rho^* > |z|$ and taken $\delta = |z|/\rho^*$. This calculation shows that $\{f_n(z)\}$ forms a Cauchy sequence and hence the limit

$$f(z) = \sum a_n z^n$$

exists provided that $|z| < 1$.

Theorem 3.1 *The function $f(z)$ is complex analytic in the open unit disk and $f'(z)$ is obtained by differentiating $f(z)$ term by term.*

Proof: Let $g_N = f - f_N$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f_N(z+h) - f_N(z)}{h} + \frac{g_N(z+h) - g_N(z)}{h}.$$

From Exercise 1 above $f_N(z)$ is complex analytic. Also, the sequence $\{na_n\}$ satisfies the UCC by Exercise 3. Hence $\lim_{N\to\infty} f'_N(z)$ exists at every point in the unit disk. Moreover, this limit is just obtained by differentiating the series for $f(z)$ term by term. To prove our result we just have to show that

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{N\to\infty} f'_N(z).$$

This is the same as showing that

$$\lim_{N\to\infty} \lim_{h\to 0} \frac{g_N(z+h)-g_N(z)}{h} = 0.$$

On the individual terms we have the bound

$$|\frac{a_n(z+h)^n - a_n z^n}{h}| = |a_n||\frac{(z+h)^n - z^n}{h}| \leq^* n|a_n||z+h|^{n-1}.$$

The starred inequality comes from the fact that the map $\phi(z) = z^n$ expands distances in $\boldsymbol{C}$ by at most $n\delta^{n-1}$ as long as $|z| \leq \delta$.

As long as h is fairly small, we can choose some $\delta < 1$ and restrict our attention to the case $|z+h| < \delta < 1$. Given the above estimate, we get

$$|\frac{g_N(z+h)-g_N(z)}{h}| \leq \sum_{n=N}^{\infty} n|a_n|\delta^{n-1} = \sum_{n=N}^{\infty} n\delta|a_n|\delta^n = R_N.$$

(We are just calling the last expression R_N for convenience.) But the sequence $\{n\delta|a_n|\}$ satisfies the UCC by Exercise 3. Hence the terms comprising R_N decay exponentially. Hence, $\lim_{N\to\infty} R_N = 0$. But the inequality above holds for any h with $|z+h| < \delta$. Hence

$$\lim_{N\to\infty} \lim_{h\to 0} |\frac{g_N(z+h)-g_N(z)}{h}| \leq \lim_{N\to\infty} R_N = 0.$$

This is what we wanted to prove. ♠

The above result, applied iteratively, shows that the kth complex derivative $f^{(k)}(z)$ is complex analytic in the unit disk and is obtained by differentiating the series for $f(z)$ k times.

In our exposition we have focused on the unit disk. Here is the obvious generalization. Say that the sequence $\{b_n\}$ satisfies the *R-convergence criterion* if the sequence $\{a_n R^n\}$ satisfies the UCC. In this case the series $\sum b_n(z-z_0)^n n$ is complex analytic in the disk of radius R about z_0 and the same result as above applies.

4 Trigonometric Interlude

It turns out that the real valued functions familiar from calculus all have extensions to $\boldsymbol{C}$ and these extensions are complex analytic. In this section you can work out the theory.

Exercise 4: Consider the Taylor series about 0 for sin() and cos(). Prove that these series satisfy the R convergence criterion for all R. Hence, the corresponding series $C(z)$ and $S(z)$ are complex analytic in $\boldsymbol{C}$. Differentiating term by term prove that $C'(z) = -S(z)$ and $S'(z) = C(z)$. Prove also that $C(0) = 1$ and $S(0) = 1$.

Let's consider the restriction of C to $\boldsymbol{R}$. In this case, the complex derivative just equals the real derivative. So, we have the differential equation $C'' = -C$ with initial conditions $C(0) = 1$ and $C'(0) = 0$. Note that cos() also satisfies this differential equation, with the same initial conditions. From the uniqueness of ordinary differential equations, we must have $C(t) = \cos(t)$. Likewise $S(t) = \sin(t)$ for $t \in \boldsymbol{R}$. So, we see that C and S are the extensions of cos() and sin() to $\boldsymbol{C}$.

If we let

$$E(z) = \sum_{n=0}^{\infty} z^n/n!$$

then we get the Taylor series for E. Differentiating term by term we get $E' = E$. Also $E(0) = 1$. Along $\boldsymbol{R}$ both $E(t)$ and e^t satisfy the same differential equation. Hence $E(t) = e^t$ for $t \in \boldsymbol{R}$.

Exercise 5: Group terms in the above series to show that

$$E(iz) = C(z) + iS(z).$$

This is a formal statement of the famous identity, due to Euler, that

$$e^{ix} = \cos(x) + i\sin(z).$$

Also, find the Taylor series for cosh() and relate $\cosh(z)$ to $\cos(iz)$.

From Exercise 5 you can see that the ordinary trig functions and the hyperbolic trig functions are really parts of the same complex analytic object. Learning about this in high school was what made me decide to be a mathematician.

5 Line Integrals an Green's Theorem

Suppose γ is an oriented smooth curve in $\boldsymbol{C}$ and f is complex analytic in a neighborhood of γ. We define a complex line integral along γ as follows. Let $g : [a, b] \to \gamma$ be a smooth parametrization.

$$\int_\gamma f\ dz = \int_a^b f(g(t))\frac{dg}{dt}\ dt.$$

Suppose that $t : [c, d] \to [a, b]$ is a diffeomorphism and $h = g \circ t$. So, $h(s) = g(t(s))$. Then $h : [c, d] \to \gamma$ is another parameterization. We have

$$\int_c^d f(h(s))\frac{dh}{hs}ds = \int_c^d f(g(t(s)))\frac{dg}{dt}\frac{dt}{ds}ds = \int_a^b f(g(t))\frac{dg}{dt}dt.$$

This is essentially the chance of variables formula from calculus. The above calculation shows that the line integral is well defined, independent of parametrization.

Exercise 6: Let λ be a counterclockwise oriented circle c entered at 0 and let $f(z) = 1/z$. Prove that $\int_\lambda f\ dz = 2\pi i$.

If we have a finite union $\Gamma = \{\gamma_j\}$ of curves, we can define

$$\int_\gamma f\ dz = \sum_j \int_{\gamma_j} f.$$

In typical applications Γ will be the union of edges of a polygon.

There is a certain symmetry which is worth remarking: Let γ^{-1} denote the oppositely oriented version of γ. Clearly we have

$$\int_\gamma f\ dz = -\int_{\gamma^{-1}} f.$$

This basic symmetry allows us to figure out complicated line integrals given simpler ones. To illustrate this, suppose that Q is a counterclockwise oriented quadrilateral and γ is a diagonal of Q, which divides Q into two triangles T_1 and T_2. If we orient these triangles counterclockwise, then γ gets a different orientation in each triangle. Hence $\int_Q f = \int_{T_1} f + \int_{T_2} f$. A similar thing happens for an N gon which has been triangulated into triangles.

Here is the basic theorem:

Theorem 5.1 (Zero Theorem) *Let γ be a simple closed loop in $\boldsymbol{C}$ and let D be the closed domain bounded by γ. Suppose that $D \subset U$ and that f is complex analytic in U then $\int_\gamma f\ dz = 0$.*

Proof: This is essentially Green's theorem in disguise. Let $f = u + iv$. Letting dx and dy be the usual line elements, we can write

$$\int_{\partial T} f\ dz = \int_{\partial T} (u+iv)(dx+idy) = \int_{\partial T} (udx - vdy) + i\int_T (vdx + udy).$$

By Green's theorem, the integral on the right equals

$$\int_T (u_y + v_x)dxdy + i\int_T (u_x - v_y)dxdy.$$

Both pieces vanish, due to the Cauchy-Riemann equations.

For the sake of giving a self-contained exposition, let me sketch a proof of the case of Green's theorem that we need. By taking limits, it suffices to prove the Zero Theorem for polygons. Any polygon can be dissected into right triangles, and so it suffices to prove the Zero Theorem for right triangles. By rotating and scaling, it suffices to prove the Zero Theorem for a right triangle with vertices 0 1 and ih. Thus, we just have to establish Green's theorem for a triangle with vertices $(0,0)$ and $(1,0)$ and $(0,h)$.

We can actually do even better than this. Both sides of Green's theorem transform in an obvious way with we apply the map $(x,y) \rightarrow (x,yh)$, and so it suffices to prove Green's theorem for the triangle with vertices $(0,0)$ $(1,0)$ and $(0,1)$. Green's theorem applies to the line element $fdx + gdy$. There is a symmetry of our triangle which interchanges the x and y axes, and so it suffices to prove Green's theorem for fdx. For this special case, the equality is just

$$\int_{\partial T} fdx = -\int_0^1 \int_0^{1-x} (f_y(x,y)\ dy)dx.$$

The line element fdx is zero along the vertical side of T. Thus, the L.H.S. is

$$\int_0^1 f(x)dx - \int_0^1 f(x, 1-x)dx.$$

The minus sign comes from taking the counterclockwise orientation of the diagonal of T. The last integral is the same as the double integral above, by the Fundamental Theorem of Calculus. ♠

6 The Cauchy Integral Formula

Suppose that γ is a simple loop bounding a domain D in which the function f is analytic. we orient γ counterclockwise. Let $a \in D$ be any point. The *Cauchy integral formula*, one of the most beautiful equations in mathematics, says that

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Here is the proof. Considering the function $f(z-a)$ in place of $f(z)$, it suffices to prove the formula when $a = 0$. Let $g(z) = f(z)/z$. Then $g(z)$ is complex analytic in the region $D - \{0\}$. Let β be the loop shown in Figure 1.

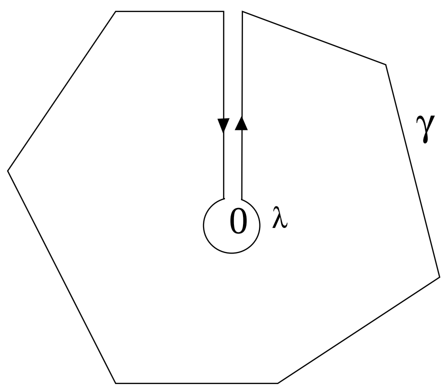

Figure 1

We have $\int_\beta g\ dz = 0$ by the Zero Theorem. The integrals over the vertical segments cancel and we get that

$$\int_\gamma g(z) = \int_\lambda g(z).$$

Here λ is a little counterclockwise circle going around 0. We have

$$\int_\gamma g(z) gz = f(0) \int_\lambda \frac{dz}{z} + \int_\lambda \frac{(f(z) - f(0)}{z}\ dz.$$

The integrand in the second integral has norm bounded by $2|f'(0)|$ as long as z is sufficiently small. Hence, the second norm of the second integral is at most $2C_\lambda |f'(0)|$ where C_λ is the circumference of λ. Letting λ shrink to a point we get (by Exercise 6)

$$\frac{1}{2\pi i} \int_\gamma g(z) = \frac{1}{2\pi i} f(0) \int_\lambda \frac{dz}{z} = f(0).$$

This is the Cauchy integral formula.

7 Infinite Differentiability

Here we will show that a complex analytic function is infinitely differentiable. To see this we use the Cauchy integral formula. It is convenient to define

$$\phi(z) = 1/(z-a).$$

By the Cauchy integral formula, and obvious properties of integration, we have

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\int_\gamma f(z)\frac{\phi(z)-\phi(z-h)}{h}dz =$$

$$\int_\gamma f(z)\lim_{h\to 0}\frac{\phi(z)-\phi(z-h)}{h}dz = -\int_\gamma f(z)\phi'(z)dz.$$

In short,

$$f'(a) = -\int_\gamma f(z)\phi'(z)dz.$$

Using ϕ' in place of ϕ and doing the same thing, we get

$$f''(a) = \int_\gamma f(z)\phi''(z)dz.$$

And so on. By induction, we get

$$f^{(n)}(a) = (-1)^n\int_\gamma f(z)\phi^{(n)}(z)dz.$$

Exercise 7: Prove the following estimate. Suppose that f is complex analytic in a neighborhood of the unit disk. Let M be the maximum value of $|f(z)|$ on the unit disk. Then $|f^{(n)}(0)| \leq 2\pi M n!$.

Exercise 7 has the following consequence. Suppose that f is complex analytic in a neighborhood of the unit disk. Then the sequence

$$\{f^{(n)}(0)/n!\}$$

is bounded and hence satisfies the UCC. In other words, the Taylor series for f about 0 defines a power series which converges and is complex analytic in the unit disk. One might ask is f equals its own Taylor series in the unit disk. This is true, and we will prove it below.

8 The Maximum Principle

Let f be a complex analytic function in U. Here we will show that f cannot take on its maximum value at a point in U, unless f is constant. We will assume that f takes on a maximum at some point $a \in U$ and we will derive a contradiction. By translating and dilating we can assume that $|f|$ takes on its maximum M at 0 and U contains the unit disk. Let γ be the circle of radius $r < 1$ centered at 0. By the Cauchy integral formula we have

$$M = |f(0)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(z)}{z}\right| \leq^* \frac{1}{2\pi r}\int_\gamma \left|\frac{f(z)}{z}\right| ds = M.$$

The starred inequality is essentially the triangle inequality. This inequality is strict unless $|f(z)| = M$ for all z on the circle of radius r. But r is arbitrary. Hence $|f(z)|$ is constant. In other words, f maps the entire unit disk on to the circle of radius M. But f is complex analytic and this means, geometrically, that *df* is a dilation followed by a rotation. Since the image of f is a single circle, this situation is only possible if f is the constant map.

For later purposes we work out some consequences of the maximum principle:

Lemma 8.1 *Suppose that $f(z)/z^n$ is complex analytic in the a neighborhood of the unit disk. Then $f(z) \leq M|z|^n$, where M is the maximum value of $|f(z)|$ on the unit circle.*

Proof: From the maximum principle we have $|f(z)|/|z^n| \leq M$. Hence $|f(z)| \leq M|z|^n$. ♠

Lemma 8.2 *Suppose, for all n, that the function $f(z)/z^n$ is complex analytic in a neighborhood of the unit disk. Then f is identically 0 on the unit disk.*

Proof: From the preceding result we have $|f(z)| \leq M|z|^n$. If $|z| < 1$ then

$$\lim_{n\to\infty} M|z|^n = 0.$$

Hence $|f(z)| = 0$ if $|z| < 1$. By continuity, $|f(z)| = 0$ if $|z| \leq 1$. ♠

9 Removable Singularities

Here we will prove the following result.

Theorem 9.1 *Suppose that f is complex analytic in a domain of the form $U - \{b\}$ and f is bounded on $U - \{b\}$. Then $f(b)$ can be (uniquely) defined so that f is complex analytic in U.*

Proof: Translating, we can assume that $b = 0$. Let γ and β and λ be the loops used to prove the Cauchy integral formula. Let $|\lambda|$ denote the radius of λ. Consider the integral

$$g(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

This function is complex analytic in the region D bounded by γ. Since $f(z)$ is bounded in a neighborhood of 0 we have

$$\lim_{|\lambda| \to 0} \int_\lambda \frac{f(z)}{z-a} dz = 0.$$

But, by the Cauchy integral formula,

$$f(a) = \frac{1}{2\pi i} \int_\beta \frac{f(z)}{z-a} dz$$

no matter which choice of λ we make. Therefore

$$f(a) = \lim_{|\lambda| \to 0} \frac{1}{2\pi i} \int_\beta \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = g(a).$$

So $g(a)$ is defined and complex analytic in all of U and $g(a) = f(a)$ as long as $a \neq 0$. So, we define $f(0) = g(0)$ and we are done. ♠

Corollary 9.2 *Suppose that f is complex analytic in a neighborhood of the unit disk and $|f(z)|/z^n$ is bounded on the unit disk for all n. Then f is identically* 0*.*

Proof: The function $f(z)/z^n$ is complex analytic in the unit disk, by the above result. Hence, the lemma from the previous section applies. ♠

10 Zero Taylor Series

In this section we prove

Theorem 10.1 *Suppose that f is complex analytic in a neighborhood of the unit disk, and the Taylor series for f is identically 0. Then f is identically 0 in the unit disk.*

Proof: If g is any function with $g(0) = 0$ we have

$$|g(z)| \leq \int_0^1 |g'(tz)|dt.$$

This is best seen geometrically, $|g'(tz)|$ measures the speed of the curve $t \to g(tz)$ which connects 0 to $g(z)$. Let's call this the *speed inequality*.

Fix n for the moment. Since $f^n(0) = 0$ we can choose $\delta > 0$ so that $|f^{(n)}(z)| < 1$ for all $|z| < \delta$. From the speed inequality we get the bound

$$|f^{(n-1)}(z)| \leq |z|$$

for all

$$|z| \leq \delta.$$

But then the speed inequality applies again and we get

$$f^{(n-2)} \leq |z|^2/2.$$

By induction we get

$$|f(z)| \leq |z|^n/n!$$

for all $|z| \leq \delta$.

Here is a technical point. Our choice of δ depends on n. If δ was independent of n we would have $|f(z)| = 0$ for $|z| < \delta$. However, this couldn't possibly work, because there are nonzero smooth functions which have 0 Taylor series.

Even though δ depends on n, the above estimate shows that $f(z)/z^n$ is bounded on the unit disk. The only thing we have to worry about is when $z \to 0$. However, our estimate takes care of this situation. So, once again, $f(z)/z^n$ is bounded on the unit disk, for any choice of n. The corollary from the preceding section now shows that f is identically 0 on the unit disk. ♠

11 Taylor Series Revisited

Suppose that f is complex analytic in a neighborhood of the unit disk. Then the Taylor series of f about 0 satisfies the UCC. Let g be the complex analytic function which equals this series in the unit disk. Since g is complex analytic, the Taylor series of g is the same as the Taylor series of f. Hence $h = f - g$ has a zero Taylor series. Hence h is identically zero in the unit disk. Hence $f = g$ in the unit disk. In short, f equals its own Taylor series in the unit disk.

Exercise 8: Consider the function

$$f(t) = \exp(-1/t^2); \qquad t > 0.$$

When $t \leq 0$ we define $f(t) = 0$. Prove that f is smooth and has a zero Taylor series about 0. This shows that smooth functions need not equal their power series.

12 Liouville's Theorem

Here is Liouville's theorem:

Theorem 12.1 *Let f be complex analytic on all of $\boldsymbol{C}$. If f is bounded then f is constant.*

Proof: Assume f is not constant. Consider the function $g(z) = f(1/z)$. This function is complex analytic in $\boldsymbol{C} - \{0\}$. Since $g(z)$ is bounded on $\boldsymbol{C} - \{0\}$, this function is complex analytic on $\boldsymbol{C}$. In particular $L = \lim_{z\to\infty} f(z) = g(0)$ exists. Since f cannot take an interior max on $\boldsymbol{C}$, we must have $|f(z)| < |L|$ on $\boldsymbol{C}$. But then 0 is an interior max for g. This is a contradiction. ♠

Exercise 9: (Challenge) Suppose that f is complex analytic in $\boldsymbol{C}$ and there is some n such that $f(z)/z^n$ is bounded. Prove that f is a polynomial of degree at most n.

We can use Liouville's theorem to give a quick proof of the fundamental theorem of algebra. Suppose that $P(z)$ is a non-constant polynomial with no roots in $\boldsymbol{C}$. Then $f(z) = 1/P(z)$ is complex analytic in all of of $\boldsymbol{C}$. Note that $\lim_{z\to\infty} |P(z)| = \infty$. Hence $f(z)$ is bounded in $\boldsymbol{C}$. Hence f is constant. Hence P is constant.

13 Disk Rigidity

Say that a *holomorphism* from one open set U to another open set V is a complex analytic homeomorphism $f : U \to V$ whose inverse image is also a complex analytic homeomorphism.

Theorem 13.1 *Let Δ be the open unit disk. Let $f : \Delta \to \Delta$ be a holomorphism. Then f is a linear fractional transformation.*

Proof: If $f(0) \neq 0$ then we can find a linear fractional holomorphism H of Δ such that $f \circ H(0) = 0$. Thus, it suffices to consider the case when $f(0) = 0$. Since $f'(0)$ exists, the function $g(z) = f(z)/z$ is bounded in Δ. Hence, this function is complex analytic. Hence $g(z)$ obeys the maximum principle.

Let C_r be the circle of radius $r < 1$ about 0. Then $|g(z)| \leq 1/r$ on C_r. Hence $|g(z)| \leq 1/r$ if $|z| < r$ by the maximum principle. Letting $r \to 1$ we see that $|g(z)| \leq 1$ on Δ. But then $|f(z)| \leq |z|$ on Δ. One the other hand $|f^{-1}(z)| \leq |z|$ by the same argument. These two inequalities force $|f(z)| = |z|$. But then $g(\Delta)$ is contained in the unit circle, a 1-dimensional curve. This is impossible unless g is a constant map. Hence there is a constant C such that $f(z) = Cz$. Hence f is a linear fractional transformation. ♠

The above result can be stated in a more compelling way. Recall that the hyperbolic isometries of Δ are exactly the linear fractional transformations which preserve Δ. Therefore, any holomorphism of Δ is actually a hyperbolic isometry.

Exercise 10 (Challenge) Let $f : \Delta \to \Delta$ be a complex analytic map which is not necessarily a holomorphism. Prove that f cannot increase hyperbolic distances. In other words $d(f(p), f(q)) \leq d(p, q)$ for all $p, q \in \Delta$.