Math 181 Handout 12

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The purpose of this handout is to summarize the material on Riemann surfaces and the Poincare Uniformization Theorem I discussed in class. Unlike previous handouts, this one doesn't have exercises. For a complete proof of the Uniformization Theorem, see Alan Beardon's book, *A Primer of Riemann Surfaces*. (L.M.S. Lecture Series **78**)

1 Riemann Surfaces

Let S be a surface. Recall that a smooth structure on S is a maximal collection of coordinate charts which have the property that the overlap functions are all smooth. A Riemann surface is defined in a similar way, with the word *complex analytic* replacing the word *smooth*. That is, a Riemann surface structure on a surface is a maximal collection of coordinate charts such that the overlap functions are all smooth. Here are some examples:

Open Subsets of $\boldsymbol{C}$ Any open subset of $\boldsymbol{C}$ is a Riemann surface. We can take the coordinate chart maps to be the identity.

The Riemann Sphere: We can think of S^2 as $\boldsymbol{C} \cup \infty$. Then $U_1 = \boldsymbol{C}$ is a neighborhood of $\{0\}$ and $U_2 = \boldsymbol{C} \cup \infty - \{0\}$ is a neighborhood of ∞. The identity map is a homeomorphism from U_1 to $\boldsymbol{C}$ and the map $f(z) = 1/z$ is a homeomorphism from U_2 to $\boldsymbol{C}$. The overlap $U_1 \cap U_2$ is $\boldsymbol{C} - \{0\}$ and the overlap function is just $f(z) = 1/z$, a complex analytic function. We already have a collection of (two) coordinate charts which cover S^2 and we can complete this collection to a maximal collection. This makes S^2 into a Riemann surface. This surface is known as *the Riemann sphere*.

Flat Tori: Let P be a parallelogram. If we glue the opposite sides of P together by translations then we produce a closed surface. Just as in class we can find a covering of S by coordinate charts whose overlap functions are translations–i.e. maps of the form $z \to z + C$ for various choices of the constant C. Such maps are complex analytic and so we can make these flat tori into Riemann surfaces in a natural way.

Hyperbolic Surfaces: Recall that a hyperbolic structure on a surface is a maximal collection of coordinate charts into $\boldsymbol{H}^2$ such that the overlap functions are all restrictions of hyperbolic isometries. If we only use orientation preserving hyperbolic isometries then these maps are all linear fractional transformations. Linear fractional transformations are complex analytic, and so a hyperbolic structure on a surface is always a Riemann surface structure.

Covering Surfaces of Riemann Surfaces Let S be a Riemann surface and let $\widetilde{S}$ be a covering of S. This means that there is a map $E : \widetilde{S} \to S$ such that each point $p \in S$ has a neighborhood U_p with the following property: The restriction of E to each component of $E^{-1}(U_p)$ is a homeomorphism. We can define coordinate charts on $\widetilde{S}$ as follows. Let $\widetilde{p} \in \widetilde{S}$ be a point. Let $p = E(\widetilde{p})$. Let U_p be as above. Then there is some component $\widetilde{U}_p$ of $E^{-1}(U_p)$ which contains $\widetilde{p}$. If $\phi : U_p \to \boldsymbol{C}$ is a coordinate chart then we can use $\phi \circ E : \widetilde{U}_p \to \boldsymbol{C}$ as a coordinate chart. We then complete this collection of coordinate charts to a maximal one. This makes $\widetilde{S}$ into a Riemann surface.

Complex Varieties: Suppose $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ is a map which is complex analytic in each coordinate. We can take the complex gradient

$$\nabla f = (\partial f/\partial z_1, \partial f/\partial z_2).$$

If ∇f does not vanish anywhere on the set $f^{-1}(0)$ then a version of the Implicit Function Theorem shows that $f^{-1}(0)$ is a Riemann surface. To make this really work well, one usually works in $\boldsymbol{CP}^2$ rather than $\boldsymbol{C}^2$. Here $\boldsymbol{CP}^2$ is the space of complex lines through the origin in $\boldsymbol{C}^3$ in the same way that the projective plane $\boldsymbol{RP}^2$ is the space of real line through the origin in $\boldsymbol{R}^3$. In natural way we have $\boldsymbol{C}^2 \subset \boldsymbol{CP}^2$ just as we have $\boldsymbol{R}^2 \subset \boldsymbol{RP}^2$ in a natural way. Namely, the point (z, w) corresponds to the complex line consisting of points (uz, uw, u), with $u \in \boldsymbol{C}$.

2 Maps between Riemann Surfaces

Suppose that S_1 and S_2 are two Riemann surfaces. A map $f : S_1 \to S_2$ is *complex analytic* in a neighborhood of $p_1 \in S_1$ if there are neighboroods U_1 of p_1 and U_2 of $p_2 = f(p_1)$, together with coordinate charts $f_j : U_j \to \boldsymbol{C}$ such that the map $f_2 \circ f \circ f_1^{-1}$ is complex analytic. f is complex analytic on S_1 if f is complex analytic in a sufficiently small neighborhood of every point. We can use some of the machinery from handout 10 to prove nontrivial results about maps between Riemann surfaces. This handout contains a sampler of these kinds of results.

Theorem 2.1 *There is no nontrivial complex analytic map from a compact Riemann surface into* $\boldsymbol{C}$.

Proof: Suppose $f : S \to \boldsymbol{C}$ is complex analytic. Since S is compact f achieves its maximum at some point $p \in S$. Let U be a coordinate chart about p and let $g : U \to C$ be a coordinate chart. Then $h = f \circ g^{-1}$ is a complex analytic map from the open set $g(U)$ into $\boldsymbol{C}$. Moreover, h takes its maximum value at an interior point of $g(U)$. But a non-constant complex analytic map cannot have an interior maximum. ♠

On the other hand, there are plenty of complex analytic maps from the Riemann sphere to itself. For instance, any rational function $R(z) = \frac{P(z)}{Q(z)}$ is a complex analytic map from the Riemann sphere to itself. Here P and Q are polynomials. The set $R^{-1}(\infty)$ is contained in the set of zeros of Q.

Theorem 2.2 *There is no nonconstant complex analytic map from* $\boldsymbol{C}$ *into a hyperbolic surface.*

Proof: Let $f : \boldsymbol{C} \to S$ be a complex analytic map from $\boldsymbol{C}$ to S. Let $E : \boldsymbol{H}^2 \to S$ be the universal covering map. Using the lifting property for maps we can find a lifting $\widetilde{f} : \boldsymbol{C} \to \boldsymbol{H}^2$ such that $E \circ \widetilde{f} = f$. (We produce $\widetilde{f}$ by partitioning $\boldsymbol{C}$ into an infinite grid of squares, and applying the lifting theorem one square at a time.) By construction $\widetilde{f}$ is complex analytic. The point is that on small neighborhoods E^{-1} is defined and complex analytic; and $\widetilde{f} = f \circ E^{-1}$ on these small neighborhoods. However we can take $\boldsymbol{H}^2$ as the open unit disk. So, $\widetilde{f}$ is a bounded complex analytic function on $\boldsymbol{C}$. However, all such maps are constant. Since $\widetilde{f}$ is constant, so is f. ♠

Theorem 2.3 *Suppose that $f : \boldsymbol{C} \to \boldsymbol{C}$ is a holomorphism. Then f is a linear map. That is $f(z) = Az + B$.*

Proof: Step 1: Subtracting off a constant we can assume that $f(0) = 0$. We claim that $f'(0) \neq 0$. Otherwise there is some $n > 1$ such that

$$f(z) = Cz^n + \text{higher.order.terms}.$$

But then f maps a little circle around 0 to a curve which winds $n > 1$ times around 0. Such a curve cannot be embedded, as it must be if f is one to one. This contradiction shows that $f'(0) \neq 0$. Hence $f(z)/z$ converges to a nonzero constant as $z \to 0$. But then $z/f(z)$ also converges to a nonzero constant as $z \to 0$.

Step 2: Since f^{-1} is continuous there is some N such that $|z| < 1$ implies that $|f^{-1}(z)| < N$. Put another way, there is some N such that $|f(z)| > 1$ provided that $|z| > N$. Consider the map

$$g(z) = 1/f(1/z).$$

Then g is complex analytic in $\boldsymbol{C} - \{0\}$ and $|g(z)| < 1$ provided that $|z| < 1/N$. Hence g is bounded on the disk of radius $1/N$ about 0. Hence g extends to a complex analytic function of all of $\boldsymbol{C}$.

Step 3 Recall from Step 1 that $z/f(z)$ is bounded in a neighborhood of 0. Using this fact, we see that $g(z)/z$ is bounded in a neighborhood of ∞. Now we know that g is complex analytic in all of $\boldsymbol{C}$ and $|g(z)| < C|z|$ as long as z is sufficiently laege. We have the Cauchy integral formula for the second derivative:

$$g''(A) = C\int_\gamma \frac{g(z)dz}{(z-A)^3},$$

for some constant C I'm too lazy to write down. We can take γ to be a circle of radius R about A. If R is large then we can use the fact that $|g(z)| < CR$ to see that the integral on the right is comparable to $1/R$. Letting $R \to \infty$ we get that $g''(A) = 0$. But A is arbitrary. Hence g'' is identically 0. But then g is linear. But then f is a linear fractional transformation. Since $f(\boldsymbol{C}) = \boldsymbol{C}$ we must have $f = Az + B$. ♠

Corollary 2.4 *Suppose that S is a Riemann surface which has a non-abelian fundamental group. Then there is no complex analytic covering map of the form $E : \boldsymbol{C} \to S$.*

Proof: Suppose that $E : \boldsymbol{C} \to S$ exists. Let G be the fundamental group of S. Then G acts on $\boldsymbol{C}$ as the deck group. Hence each $g \in G$ must act as a linear map on $\boldsymbol{C}$. Also, g does not fix and points of $\boldsymbol{C}$ because a deck transformation is the identity if it fixes one point. The only linear maps with this property are the translation. In short g is a translations. But any two translations commute and hence G is abelian. This contradiction shows that E does not exist. ♠

3 The Uniformization Theorem

The Uniformization Theorem can be viewed as an extension of the classical Riemann mapping theorem. The classical Riemann theorem is this:

Theorem 3.1 *Let A be a simply connected open subset of $\boldsymbol{C}$. Then there is a holomorphism between A and the open unit disk.*

Proof: (Sketch) Say that a *rectilinear region* is an open subset of $\boldsymbol{C}$ bounded by a polygon whose sides are parallel to the coordinate axes. In the previous handout I showed how to use the Schwarz-Christoffel transform to get holomorphisms between the unit disk and many rectilinear regions. Once the sequence of left and right turns of the region is fixed, the side lengths are determined by the placement of the special points on the boundary of the upper half plane. Using the Brouwer fixed point theorem (a higher dimensional analog of the intermediate value theorem) we can prescribe the side lengths arbitrarily. In other words, we can find a holomorphism between the unit disk and any rectilinear region. Finally, any simply connected region can be approximated by a sequence of rfectilinear regions, and the associated sequence of maps, if suitably normalized, converges to the desired final map. ♠

Here is a version of the Uniformization Theorem, sometimes also called the Riemann mapping theorem:

Theorem 3.2 *Let A be and simply connected Riemann surface. Then one of three things is true:*

- *A is compact, and there is a holomorphism between A and the Riemann sphere.*
- *A is non-compact and there is a holomorphism between A and $\boldsymbol{C}$.*
- *A is non-compact and there is a holomorphism between A and the open unit disk.*

Combining the Uniformization Theorem with the Corollary above, we get:

Lemma 3.3 *There is a complex analytic covering map from the open unit disk to $\boldsymbol{C} - 0 - 1$, the twice punctured plane.*

Proof: The universal cover X of $\boldsymbol{C} - 0 - 1$ is a simply connected Riemann surface. Let $E : X \to \boldsymbol{C} - 0 - 1$ be the covering map. If X is compact then $E(X)$ is also compact, since the image of a compact set under a continuous map is conpact. But $E(X) = \boldsymbol{C} - 0 - 1$, which is noncompact. So, X is noncompact. If there is a holomorphism between X and $\boldsymbol{C}$ then we have a complex analytic cover $\boldsymbol{C} \to \boldsymbol{C} - 0 - 1$. However, the fundamental group of $\boldsymbol{C} - 0 - 1$ is non-abelian. This is a contradiction. We have only one alternative left in the Uniformization Theorem,, and so there is a holomorphism h between X and the open unit disk. But then $E \circ h^{-1}$ is the desired complex analytic covering map between the open unit disk and $\boldsymbol{C} - 0 - 1$. ♠

Now we have all the machinery to prove the famous Picard Theorem:

Theorem 3.4 *Let $f : \boldsymbol{C} \to \boldsymbol{C}$ be a non-constant analytic map. Then either f is onto or f omits exactly one value.*

Proof: We will suppose that f omits at least two values and show that f is constant. We can scale f so that two of the omitted values are 0 and 1. Then $f : \boldsymbol{C} \to \boldsymbol{C} - 0 - 1$. We have our holomorphic covering from the open unit disk Δ to $\boldsymbol{C} - 0 - 1$. But then we can find a lift $\widetilde{f} : \boldsymbol{C} \to \Delta$. This map is a bounded complex analytic function, and hence constant. Hence f is constant as well. ♠

4 Implications for Compact Surfaces

The Uniformization Theorem is stated above in terms of simply connected Riemann surfaces, but it has nice implications for general surfaces. To get a clean statement, I'll stick to the case of compact oriented surfaces.

Theorem 4.1 *Let S be a compact and oriented Riemann surface. This is to say that S is homeomorphic to either the sphere, the torus, or a higher genus surface. Then the following is true about S.*

- *If S is homeomorphic to a sphere, then there is a holomorphism between S and the Riemann sphere.*
- *If S is homeomorphic to the torus, then there is a holomorphism between S and a flat torus.*
- *If S is homeomorphic to a higher genus surface, then there is a holomorphism between S and some hyperbolic surface.*

Proof: If S is homeomorphic to the sphere then the Uniformization Theorem immediately says that there is a holomorphism from S to the Riemann sphere.

Suppose that S is not homeomorphic to a torus. Then the fundamental group of S is nonabelian. By the Uniformization Theorem and the corollary above, we have a complex analytic covering $\Delta \to S$ where Δ is the unit disk. Let G be the fundamental group of S. Then G acts on Δ as the deck group. Each element $g \in G$ is a holomorphism of Δ. In handout 10 we proved that such maps are hyperbolic isometries. Hence G acts on Δ as a group of hyperbolic isometries. S is precisely the quotient of the hyperbolic plane by the orbit equivalence relation: Two points are equivalent iff there is some element of G which maps one to the other. Small neighborhoods of points in Δ contain unique members of equivalence classes, and so these little disks map injectively into S. The inverse maps give local coordinate charts into Δ, such that the overlap functions are restrictions of hyperbolic isometries. In short, S inherits its hyperbolic structure from Δ.

Suppose that S is homeomorphic to a torus. If there is a holomorphic covering $\Delta \to S$ then the same argument as just given shows that S is a hyperbolic surface and the fundamental group $\boldsymbol{Z}^2$ acts on Δ by hyperbolic isometries. This is only possible if all the elements of $\boldsymbol{Z}^2$ fix a common point on the unit circle. Such maps have the following property: For any $\epsilon > 0$

there is some point $x \in \Delta$ which is moved less than ϵ (as measured in the hyperbolic metric.) But then S would have closed and homotopically nontrivial loop of length less than ϵ. This contradicts the fact that all sufficiently short loops on S are homotopically trivial. The contradiction shows that there is no holomorphic cover from Δ to S. Only one alternative for the Uniformization Theorem holds and so there is a holomorphic cover $\boldsymbol{C} \to S$. But now the deck transformations are all Euclidean translations and S inherits a Euclidean structure from $\boldsymbol{C}$ just as in the previous case. ♠

The above theorem in much more generality. For instance, suppose that $C \subset \boldsymbol{C}$ is a finite set of $N > 2$ points. Then there is a holomorphism between $\boldsymbol{C} - C$ and a hyperbolic surface. The same result holds if C is a countably infinite set of points, or the middle-third Cantor set. It's hard to picture the universal cover of the complement of the middle-third Cantor set, but the Uniformization Theorem says that it is just the hyperbolic plane in disguise.