Math 181 Handout 3

Rich Schwartz

October 4, 2005

The purpose of this handout is to compute the fundamental group for some familiar objects:

- the circle;
- the torus;
- the 2-sphere;
- the projective plane.
- lens spaces the Poincare sphere.

I'll work out the first three in detail and then somewhat guide you through the computation for the others. The last section is too advanced for an undergraduate course but I couldn't resist.

1 The Circle

1.1 A Special Map

Let S^1 be the circle. We think of S^1 as the group of unit complex numbers in $\boldsymbol{C}$. Let $\boldsymbol{R}$ denote the real numbers. There is a natural map $E : \boldsymbol{R} \to S^1$ given by

$$E(t) = \exp(2\pi i t) = \cos(2\pi t) + i\sin(2\pi t).$$

This map is certainly onto and continuous, but it has some other special properties. Say that an *open special arc* in S^1 is a set of the form

$$C(z) = \{w \in S^1 |\ d(z, w) < 1/100\}.$$

Here $d(z,w) = |z-w|$, the usual Euclidean distance. The choice of $1/100$ is convenient but fairly arbitrary. The point is just that open special arcs are much smaller than semicircles.

Exercise 1: Let C be an open special arc. Prove that $E^{-1}(C)$ consists of a countably infinite number of disjoint open intervals and that the restriction of E to any of these intervals is a homeomorphism from the interval onto C.

Based on this exercise we're going to establish two results which are part of a general theorem. We'll get to the general result later on in the course.

Lemma 1.1 *Let $[a,b] \subset \boldsymbol{R}$ be a closed interval. Suppose that $g : [a,b] \to S^1$ is a map such that $g([a,b])$ is contained in a special arc. Suppose also that there is a map $\widetilde{g} : \{a\} \to \boldsymbol{R}$ such that $E \circ \widetilde{g}(a) = g(a)$. Then we can define $\widetilde{g} : [a,b] \to \boldsymbol{R}$ such that $E \circ \widetilde{g} = g$ on all of $[a,b]$. This extension of $\widetilde{g}$ is unique.*

Proof: If E had an inverse we could define $\widetilde{g} = E^{-1} \circ g$. Also, we would be forced to make this definition and so the extension of $\widetilde{g}$ to $[a,b]$ would be unique. Unfortunately, E is not invertible. Fortunately, we have Exercise 1, which shows that E is "invertible" in some sense. Let C be the special arc which exists by hypothesis. Let $\widetilde{C} \subset E^{-1}(C)$ be the unique interval from Exercise 1 which contains $\widetilde{g}(a)$. By Exercise 1, the map $E : \widetilde{C} \to C$ is a homeomorphism. Let $F : C \to \widetilde{C}$ be the inverse of (the restricted version of) E. Since $g[a,b] \subset C$ we can (and must) define $\widetilde{G} = F \circ g$. ♠

Let $1 = E(\boldsymbol{Z})$ be the basepoint of S^1. Let $I = [0,1]$. Recall that an element of $\pi_1(S^1, 1)$ is a map $g : I \to S^1$ such that $g(0) = g(1) = 1$.

Exercise 2: Given the map g, prove that there exists some N with the following property. If $x, y \in [0,1]$ and $|x-y| < 1/N$ then the set $g([x,y])$ is contained in a special arc. Hint: You might want to use the fact that every infinite sequence in $[0,1]$ has a convergent subsequence. This is basically the Bolzano–Weierstrass theorem

1.2 The Winding Number

Here is an improved version of Lemma 1.1.

Lemma 1.2 *Let* $g : [0,1] \to S^1$ *be a loop. Then there is a unique map* $\widetilde{g} : [0,1] \to \boldsymbol{R}$ *such that* $\widetilde{g}(0) = 0$ *and* $E \circ \widetilde{G} = G$ *on all of* $[0,1]$.

Proof: From Exercise 2 we can find some N such that the points $t_i = i/N$ have the following property: The image $g([t_i, t_{i+1}])$ is contained in a special arc for $i = 0, ..., N-1$. Now we go by induction. First of all, by Lemma 1.1 we can define $\widetilde{g}$ uniquely on $[t_0, t_1]$. But then by Lemma 1.1 again we can define $\widetilde{g}$ uniquely on $[t_1, t_2]$. And so on. ♠

Note that the above definition doesn't depend on N. If we chose some larger N' and redid the definition, the uniqueness over small intervals would tell us that the two $\widetilde{g}$ maps would agree everywhere.

We define the *winding number* of g to be the value of $\widetilde{g}(1) \in \boldsymbol{Z}$. We write this as $w(g)$. Note that $\widetilde{g}(1) \in \boldsymbol{Z}$ because $g(1) = E(\widetilde{g}(1)) = 0$.

1.3 Invariance Properties

Now suppose that $[g_0] = [g_1] \in \pi_1(S^1, 1)$. This means that g_0 and g_1 are loop homotopic to each other. We want to prove that $w(g_0) = w(g_1)$. Let $g_t(x) = G(x,t)$. We would like to say that $w(g_t)$ is a continuous and integer valued map, hence constant.

The same argument as in Exercise 2 proves the following result: There is some N with the following property: If $s, t \in [0,1]$ are any points such that $|s - t| < 1/N$ and $x \in [0,1]$ is fixed then $|G(x,s) - G(x,t)| < 1/100$. Using the other notation, $d(g_s(x), g_t(x)) < 1/100$. But then $d(\widetilde{g}_s(x), \widetilde{g}_t(x))$ is either less than $1/100$ or greater than $1/2$. By continuity, the alternative can't change. Also

$$d(\widetilde{g}_s(0), \widetilde{g}_t(0)) = d(0,0) = 0 < 1/100.$$

This shows that the first alternative always holds and $\widetilde{g}_s(x)$ and $\widetilde{g}_t(x)$ are always within $1/100$ of each other. But then $w(g_s) = w(g_t)$, because both are integers within $1/100$ of each other. From here it is easy to see that $w(g_0) = w(g_1)$.

Now we can define

$$w([g]) = w(g).$$

In other words, the winding number of an element of $\pi_1(S^1, 1)$ is well defined. This gives us a map $w : S^1 \to \boldsymbol{Z}$.

Execise 3: Prove that w is an onto homomorphism.

1.4 Injectivity

To show that w is an isomorphism we need to show that the kernel of w is trivial. That is, if $[g] \in \pi_1(S^1, 1)$ is such that $w([g]) = 0$ then $[g] = 0$. In other words, if $w(g) = 0$ then g is loop homotopic to the constant loop. If $w(g) = 0$ then $\widetilde{g} : [0, 1] \to \boldsymbol{R}$ is a loop. But $\pi_1(\boldsymbol{R}, 0) = 0$. Hence there is a loop homotopy $\widetilde{G}$ from $\widetilde{g}$ to the constant loop $\widetilde{g}_0 : S^1 \to \boldsymbol{R}$. But then $E \circ \widetilde{G}$ is a loop homotopy from g to the constant loop in S^1. This shows that w is an isomorphism.

Hence $\pi_1(S^1, 1) = \boldsymbol{Z}$.

1.5 An Abstract Framework

Notice that the main property we used about the circle was the existence and special properties of the map $E : \boldsymbol{R} \to S^1$. We also used the property that $\pi_1(\boldsymbol{R}, 0) = 0$. It turns out that this will be a general method for us when we compute the fundamental groups. All the special properties we established are summarized by the statement that $\boldsymbol{R}$ is the *universal cover* of S^1 and E is the *universal covering map*. In the next handout I'll develop these ideas in great generality. We'll see these ideas again below.

2 The Torus

It was a previous (easy) exercise to show that

$$\pi_1((Y, y) \times (Z, z)) = \pi_1(Y, y) \times \pi_1(Z, z).$$

The torus T^2 is homeomorphic to $S^1 \times S^1$ and also path connected. Hence $\pi_1(T^2) = \boldsymbol{Z} \times \boldsymbol{Z}$. Iterating, we get $\pi_1(T^n) = \boldsymbol{Z}^n$.

3 The 2-Sphere

Let $I = [0,1]$ as above. Let $x \in S^2$ be some basepoint. Say that a loop $g : I \to S^2$ (anchored at x) is *bad* if $g(I) = S^2$ and otherwise good. We have already seen in class that any good loop is loop homotopic to the constant loop. In this section we'll show that any bad loop is loop homotopic to a good loop. Hence $\pi_1(S^1, x) = 0$.

Exercise 4: Let $[a,b]$ be an interval and let Δ be the open unit disk in $\boldsymbol{R}^2$. Let $f : [a,b] \to \Delta$ be a continuous map. Prove that there is homotopy $F : [a,b] \times [0,1] \to \Delta$ such that

- $F(a,t)$ and $F(b,t)$ are independent of t.
- $F(x,0) = f(x)$ for all x.
- $f_1 : [a,b] \to \Delta$ is a linear map. Here $f_1(x) = F(x,1)$.

Intuitively, F straightens out $f([a,b])$. You should think of a rubber band tortured into a complicated position and then (except for the endpoints) relased.

Exercise 5: Let H be an open hemisphere of S^2. Prove that there is a homeomorphism from Δ to H which maps straight lines to circular arcs.

Combining the last two exercises we get the following result: Let $[a,b]$ be an interval and let H be a hemisphere in $\boldsymbol{R}^2$. Let $f : [a,b] \to H$ be a continuous map. Then there is homotopy $F : [a,b] \times [0,1] \to H$ such that

- $F(a,t)$ and $F(b,t)$ are independent of t.
- $F(x,0) = f(x)$ for all x.
- $f_1[a,b]$ is contained in a circular arc.

Exercise 6: Prove that any loop $f : I \to S^2$ is loop homotopic to a loop which is contained in a finite union of circles. Hint: combine something like Exercise 2 with the straightening procedure.

A finite union of circles cannot cover S^2 and so any loop, including a bad one, is loop homotopic to a good loop. Hence $\pi_1(S^2, x) = 0$.

4 The Projective Plane

Think of $\boldsymbol{P}^2$, the projective plane, as the quotient $S^2/\sim$, where $x \in S^2$ is equivalent to itself and to the antipodal point $-x$. There is a nice map $E : S^2 \to \boldsymbol{P}^2$ given by $E(x) = [x]$. (Remember E from above? Here is a reincarnation.)

Let $x_+ = (0, 0, 1)$ and let $x_- = (0, 0, -1)$. Note that $E(x_+) = E(x_-)$.

Exercise 7: Suppose that $g : [0, 1] \to \boldsymbol{P}^2$ is a loop based at x_+. Prove that there is a unique map $\widetilde{g} : [0, 1] \to S^2$ such that $\widetilde{g}(0) = x_+$ and $E \circ \widetilde{g} = g$ Hint: Just imitate what was done for the circle.

Note that either $\widetilde{g}(1) = x_+$ or $\widetilde{g}(1) = x_-$. We define $w(g) = +1$ if $\widetilde{g}(1) = x_+$ and $w(g) = -1$ if $\widetilde{g}(1) = x_-$.

Exercise 8: Prove that $w([g])$ is well defined independent of the loop homotopy equivalence class of g. Prove also that w gives an isomorphism from $\pi_1(\boldsymbol{P}^2)$ to $\boldsymbol{Z}/2$.

In general we have $\boldsymbol{P}^n = S^n/\sim$, where $x \sim -x$. Thus there is always this two-to-one map from S^n to $\boldsymbol{P}^n$. An argument similar to the one given above shows that $\pi_1(\boldsymbol{P}^n) = \boldsymbol{Z}/2$. Here $\boldsymbol{P}^n$ is called *projective n-space*.

5 Some 3 Dimensional Examples

5.1 A Lens Space

Let's think of S^3 as the set of the form

$$\{(z, w)|\ |z|^2 + |w|^2 = 1\} \subset \boldsymbol{C}^2 = \boldsymbol{R}^4.$$

This is an exotic way of expressing the fact that S^3, the 3-sphere, is the unit sphere in $\boldsymbol{R}^4$. The equality $\boldsymbol{C}^2 = \boldsymbol{R}^4$ probably needs some explanation, because these spaces are technically not equal. However, there is an obvious real linear isomorphism between them:

$$(x_1 + iy_1, x_2 + iy_2) \to (x_1, y_1, x_2, y_2),$$

and this map is preserving of the two relevant metrics.

Here is a nice equivalence relation on S^3. Let's define

$$(z,w) \sim (uz, u^2 w)$$

if and only if u is some 5th root of unity. Each equivalence class on $S^3/\sim$ has 5 points. Let's call this space $L(2,5)$. The 2 comes from the u^2 term and the 5 comes from the fact that we are taking 5th roots of unity. Obviously, you could make this construction for other choices.

Exercise 9: Prove that $L(2,5)$ is a good quotient and also a manifold.

There is an obvious map $E: S^3 \to L(2,5)$. Using E we can show that $\pi_1(L(2,5)) = \boldsymbol{Z}/5$. Generalizing this construction in an obvious way, we see that we can produce a 3 manifold whose fundamental group is $\boldsymbol{Z}/n$. These spaces $L(m,n)$ are called *Lens spaces*.

5.2 The Poincare Manifold

Let $SO(3)$ denote the group of orientation preserving (i.e. physically possible) rotations of S^2. It turns out that there is an amazing map from S^3 to $SO(3)$ which is really the map from S^3 to $\boldsymbol{P}^3$ in disguise. So, given an element $q \in S^3$ we need to produce a rotation R_q of S^2.

Here's the construction. Let's think of S^3 as the *unit quaternions*. That is, a point in S^3 can be thought of as a symbol of the form

$$a + bi + cj + dk; \qquad a,b,c,d \in \boldsymbol{R}; \qquad a^2+b^2+c^2+d^2 = 1.$$

The symbols i,j,k satisfy the following rules

- $i^2 = j^2 = k^2 = -1$.
- $ij = k$ and $jk = i$ and $ki = j$.

Given these rules you can multiply quaternions together in a way which is similar to how you multiply complex numbers together.

Given any $q \in S^3$ as above, we define

$$q^{-1} = a - bi - cj - dk.$$

Then you can check that $qq^{-1} = q^{-1}q = 1$. In other words, the unit quaternions form a group under multiplication!

We can identify $\boldsymbol{R}^3$ with the pure quaternions, namely those of the form $0 + bi + cj + dj$. The isomorphism to $\boldsymbol{R}^3$ is just given by

$$0 + ai + bj + ck \to (a, b, c).$$

Thus our special $\boldsymbol{R}^3$ has the usual Euclidean metric on it, coming from the identification with the usual $\boldsymbol{R}^3$.

Given $p \in \boldsymbol{R}^3$ we define

$$R_q(p) = qpq^{-1}.$$

Exercise 10 (Challenge) Show that R_q preserves $\boldsymbol{R}^3$ (the pure quaternions) and is an orientation preserving rotation.

Multiplication turns out to be associative and so we have

$$R_{q_1} \circ R_{q_2}(p) = q_1(q_2 p q_2^{-1})q_1^{-1} = R_{q_1} \circ R_{q_2}(p).$$

This works for any p. Hence the map $q \to R_q$ is a homomorphism. As you might expect, we define

$$E(q) = R_q.$$

Note that $E(-q) = E(q)$. It turns out that the kernel of E is precisely $\{1, -1\}$. So, E is both a continuous surjection (with good local inverse properties) and a two-to-one homomorphism from S^3 to $SO(3)$.

Now the fun starts. If $G \subset SO(3)$ is a finite subgroup then $\widetilde{G} = E^{-1}(G)$ is a subgroup with twice the number of elements. Now we can define an equivalence on S^3 by the rule $q_1 \sim q_2$ iff there exists some $g \in \widetilde{G}$ such that $gq_1 = q_2$. If G has N elements then $\widetilde{G}$ has $2N$ elements and each equivalance class of $S^3/\sim$ has $2N$ elements. It turns out the quotient space is a manifold with fundamental group $\widetilde{G}$.

As a special case, let G be the orientation preserving symmetries of the icosahedron, the most interesting finite subgroup of $SO(3)$. Then $\widetilde{G}$ is an order 120 group known as the *binary icosahedral group.*. The quotient in this case is called the *Poincare manifold* and its fundamental group is $\widetilde{G}$.

This is one of the great examples in geometry.