Math 181 Handout 7

Rich Schwartz

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The purpose of this handout is to give you a bare bones introduction to hyperbolic geometry. The material in this handout can be found in a variery of sources, for example:

- Alan Beardon's book, *the Geometry of Discrete Groups*.
- Svetlana Katok's book, *Fuchsian Groups*.
- William Thurston's book, *The Geometry and Topology of* 3*-Manifolds*.

1 The Upper Half Plane Model

Let $U \subset \boldsymbol{C}$ be the upper halfplane, consisting of points z with $\text{Im}(z) > 0$. As in Exercise 3 of Handout 6, we define a Riemannian metric on U by the formula

$$G_z(v,w) = \frac{v \cdot w}{y^2}; \qquad y = \text{Im}(z).$$

When U is equipped with this metric, we denote it by $\boldsymbol{H}^2$ and call it *the hyperbolic plane*.

This Riemannian metric has more symmetries than you might think. We will now explore this. Suppose that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a real 2×2 matrix with determinant 1. The set of these matrices is denoted by $SL_2(\boldsymbol{R})$. In fact, this set forms a group under matrix multiplication.

The matrix A defines a *real linear fractional transformation*

$$T_A(z) = \frac{az+b}{cz+d}.$$

Note that the denominator of $T_A(z)$ is nonzero as long as $z \notin \boldsymbol{R}$. Thus $T_A(z)$ is defined and finite as long as $z \notin \boldsymbol{R}$.

Exercise 1: Prove that $z \notin \boldsymbol{R}$ implies that $T_A(z) \notin \boldsymbol{R}$. Prove also that T_A maps $\boldsymbol{H}^2$ into itself. (Hint: show that this is true for a single choice of A, then show/use the fact that $SL_2(\boldsymbol{R})$ is path connected.)

Exercise 2: Establish the general formula

$$T_{AB} = T_A \circ T_B,$$

where $A, B \in SL_2(\boldsymbol{R})$.

In light of Exercise 2, we can take $B = A^{-1}$. Then T_A and T_B are inverses of each other. Also T_A and T_B are clearly continuous. Hence, T_A is a homeomorphism which maps $\boldsymbol{H}^2$ to itself (and T_B is the inverse homeomorphism.)

Exercise 3: Say that a real linear fractional transformation is *basic* if it has one of three forms:

- $T(z) = z + 1$.
- $T(z) = rz$.
- $T(z) = -1/z$.

Prove that any real linear fractional transformation is the composition of basic ones.

Here is an argument that the map $T(z) = rz$ is a Riemannian isometry. Let $(x, y) \in \boldsymbol{H}^2$ be a point and let v, w be two vectors. Then $T(p) = (rx, ry)$ and $dT(v) = rv$ and $dT(w) = rw$. But then

$$G_{rp}(rv, rw) = \frac{rv \cdot rw}{r^2 y^2} = \frac{v \cdot w}{y^2} = G_p(v, w).$$

This proves it. A similar argument works for the map $T(z) = z + 1$.

Exercise 4: Prove that the map $T(z) = -1/z$ is a Riemannian isometry of $\boldsymbol{H}^2$.

Combining Exercises 3 and 4, we see that any real linear fractional transformation is a Riemannian isometry of $\boldsymbol{H}^2$.. Recall that we proved $SL_2(\boldsymbol{R})$ is a 3 dimensional manifold. So, $\boldsymbol{H}^2$ has a 3-dimensional group of symmetries!

2 Symmetries and Circles

Say that a *generalized circular arc* is either an arc of a circle or a segment of a straight line, or an infinite ray. The point of this definition is that an unboundedly large sequence of circular arcs can converge to a subset of a line, so we want a definition of "circular arc" which is closed under taking limits. Also, our definition interacts well with the linear fractional transformations and hyperbolic geometry.

Say that a map T from $\boldsymbol{H}^2$ to itself is *circle preserving* it maps generalized circular arcs to generalized circular arcs. That is, if A is a generalized circular arc contained in $\boldsymbol{H}^2$ then $T(A)$ is also a generalized circular arc contained in $\boldsymbol{H}^2$.

The maps $T(z) = z + 1$ and $T(z) = rz$ are obviously circle preserving.

Lemma 2.1 *The map $T(z) = -1/z$ is circle preserving.*

Proof: By continuity, it suffices to prove this result for a dense set of generalized circular arcs A. In particular, it suffices to consider the case when A is not contained in a straight line. In this case all the points $z \in A$ satisfy an equation of the form

$$|z - c|^2 = r^2.$$

The center of the circle is c and the radius is r. We can write the above equation as

$$|z|^2 - c(z + \overline{z}) + d = 0.$$

Here we have set $d = c^2 - r^2$. Any collection of points which satisfies an equation like this lies on the circle.

All the points w on $T(A)$ satisfy the equation

$$|1/w - b|^2 = s^2.$$

Here we have set $b = 1/a$ and $s = 1/r$. Expanding this out and rearranging, we find that

$$\frac{1 - b(w + \overline{w}) + b^2|w|^2}{|w|^2} = s^2.$$

This can be rearranged into

$$1 - b(w + \overline{w}) + (b^2 - s^2)|w|^2 = 0.$$

For a dense set of choices of A we have $b^2 - s^2 \neq 0$ and so we can divide through to obtain

$$|w|^2 + \frac{b}{b^2 - s^2}(w + \overline{w}) + \frac{1}{b^2 - s^2} = 0.$$

Setting

$$c' = \frac{b}{b^2 - s^2}; \qquad d' = \frac{1}{b^2 - s^2}$$

we see that

$$|w|^2 - c'(w + \overline{w}) + d' = 0.$$

This is the equation for a circle. Hence $T(A)$ is a circular arc. ♠

There are some nice geometric proofs of the above lemma. See David Hilbert's book, *geometry and the imagination*. The basic geometric reason that the above lemma works is that stereographic projection maps circular arcs on the sphere to generalized circular arcs in the plane. (Try to prove this.)

Using Exercise 3, we see that every real linear fractional transformation is circle preserving.

Exercise 5: Prove that a real linear fractional transformation T has the following property: If a and b are two smooth curves in $\boldsymbol{H}^2$ which intersect at a point x and make an angle of θ then $T(a)$ and $T(b)$ make the same angle θ at the point $T(x)$. Hint: If you don't feel like grinding out the calculation you can assume the result is false and then deduce that the differential dT fails to map circle to circles. In any case, the result is obvious for all the basic maps except $z \to -1/z$ and so it suffices to consider this one.

3 Geodesics

Say that a *geodesic* in $\boldsymbol{H}^2$ is either a ray which is perpendicular to the x-axis, or else a semicircular arc which meets the x-axis at right angles. If A is a geodesic then $T(A)$ is a generalized circular arc. Since T preserves intersection angles, $T(A)$ also meets the x-axis at right angles. Hence $T(A)$ is also a geodesic. In other words the real linear fractional transformations permute the geodesics in the same way that isometries of Euclidean space permute the straight line segments.

Exercise 6: Suppose that A_1 and A_2 are two geodesics. Prove that there is a real linear fractional transformation T such that $T(A_1) = T(A_2)$. Hint: work with the basic transformations first.

Lemma 3.1 *Let A be the geodesic which is the positive y axis. Let p and q be two points on A. Then the portion of A connecting p to q is the unique shortest curve in $\boldsymbol{H}^2$ which connects these two points.*

Proof: Consider the map F defined by the equation $F(x, y) = (0, y)$. Looking at the definition of the hyperbolic metric, we see that F is hyperbolic speed non-increasing. That is, if γ is a curve in $\boldsymbol{H}^2$ then the hyperbolic speed of $F(\gamma)$ at any point is at most the hyperbolic speed of γ at the corresponding point. Moreover, if the velocity of γ has any x-component at all then $F(\gamma)$ is slower at the corresponding point. The idea here is that F does not change the y-component of the hyperbolic speed, but kills the x-component. The total hyperbolic length of γ is the integral of its hyperbolic speed. Thus the hyperbolic length of $F(\gamma)$ is less than the hyperbolic length of γ, unless γ travels vertically the whole time. Our result follows immediately from this. ♠

Corollary 3.2 *Let A be any geodesic in $\boldsymbol{H}^2$. Let p and q be two points on A. Then the portion of A connecting p to q is the unique shortest curve in $\boldsymbol{H}^2$ which connects these two points.*

Proof: We can apply a Riemannian isometry T such that $T(A)$ is as in the previous lemma. Then we can apply the previous lemma. ♠

A *geodesic segment* is defined to be the portion of a geodesic which joins two points on that geodesic. Summarizing what we know so far:

- Every real linear fractional transformation is a hyperbolic isometry.
- Every real linear fractional transformation maps geodesics to geodesics.
- Every geodesic segment is the unique shortest curve connecting its endpoints.

The geoedesics and geodesic segments play the role, in hyperbolic geometry, that lines and line segments play in Euclidean geometry.

4 The Disk Model

Let Δ be the open unit disk. There is a nice map $M : \boldsymbol{H}^2 \to \Delta$ given by

$$M(z) = \frac{z-i}{z+i}.$$

This map does the right thing because $z \in \boldsymbol{H}^2$ is always closer to i than to $-i$ and so $|M(z)| < 1$.

Exercise 7: Prove that M maps geodesics in $\boldsymbol{H}^2$ to generalized circular arcs which meet the unit circle at right angles.

There is a unique Riemannian metric H on Δ which makes M into an isometry. By symmetry, this Riemannian metric must be a multiple of the ordinary dot product at the point $0 \in \boldsymbol{C}$. A calculation shows that

$$H_z(v,w) = \frac{4v \cdot w}{(1-|z|)^2}.$$

The factor of 4 comes from the fact that $|M'(i)| = 1/2$. Equipped with this Riemannian metric, Δ is also called the hyperbolic plane. The geodesics in Δ are the generalized circular arcs which meet the unit circle at right angles.

Exercise 8: Draw a nice picture of 10 different geodesics in the disk model of $\boldsymbol{H}^2$.

Note that M is also a linear fractional transformation, through not a real linear transformation. When T is a real linear fractional transformation, the map $M \circ T \circ M^{-1}$ is an isometry of Δ. It is useful to have both models of the hyperbolic plane available. In the next section we'll see that certain results about polygons are easy in the disk model.

5 Geodesic Polygons

Say that a *geodesic polygon* in $\boldsymbol{H}^2$ is a simple closed path made from geodesic segments. *Simple* means that the path does not intersect itself. Say that a *solid* geodesic polygon is the region in $\boldsymbol{H}^2$ bounded by a geodesic polygon. Just as in Euclidean geometry, we can measure the angles between the sides of a geodesic polygon. This works out because the hyperbolic isometries preserve angles of intersection. Here is a classical result in non-Euclidean geometry:

Lemma 5.1 *Let T be a geodesic triangle in the hyperbolic plane. Then the sum of the angles of T is less than π.*

Proof: We can work in the disk model. There are enough isometries of $\boldsymbol{H}^2$ to move any point to any other point. In particular, we can move a point in the interior of (the solid version of) T so that it is 0. But then the sides or T are circular arcs which bend inwards towards 0. (Draw a picture and you will see what I'm talking about.) ♠

A solid geodesic polygon P is *convex* if it has the following propery: If $p, q \in P$ are two points then the geodesic segment joining p and q is also contained in P. It is easy to prove, inductively, that any convex geodesic polygon can be decomposed into geodesic triangles.

Corollary 5.2 *The sum of the interior angles in a convex geodesic N gon is at most $(N-2)\pi$.*

Proof: Just decompose into triangles and then apply the previous lemma multiple times. ♠

Exercise 9: (Challenge) Suppose that $\theta_1, \theta_2, \theta_3$ are three numbers whose sum is less than π. Prove that there is a hyperbolic geodesic triangle with angles $\theta_1, \theta_2, \theta_3$. (Hint: the intermediate value theorem.)

Exercise 10: (Challenge) Say that a geodesic triangle is *δ-thin* if every point in the interior of the (solid version of) triangle is within δ of a point on the boundary. Note that there is no universal δ so that all Euclidean triangles are δ thin. Prove that all hyperbolic geodesic triangles are 10-thin. (The value $\delta = 10$ is far from optimal.)