Math 181 Handout 8

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The purpose of this handout is to define hyperbolic surfaces and to present a general method of constructing them out of convex geodesic polygons.

1 Definition

Let Σ be a surface. This to say that Σ is a metric space, and every point of Σ has a neighborhood which is homeomorphic to $\boldsymbol{R}^2$. Actually, it is a little bit annoying to always have to map the neighborhoods of Σ to all of $\boldsymbol{R}^2$. So, we define a *disklike subset* of $\boldsymbol{R}^2$ to be a subset which is homeomorphic to $\boldsymbol{R}^2$. For instance, an open disk is disklike. So is an open (solid) square. So, we can equally well say that a surface is a metric space such that every point has a neighborhood which is homeomorphic to a disklike subset of $\boldsymbol{R}^2$. If a disklike set is contained in the upper half plane, we call it a disklike subset of $\boldsymbol{H}^2$.

Let U and V be two open subset of $\boldsymbol{H}^2$. Say that a map $f : U \to V$ is a *local hyperbolic isometry* if the restriction of f to each open component of U agrees with the restriction of a hyperbolic isometry. The easiest case to think about is when U and V are both connected. Then $f : U \to V$ is a local isometry iff f is the restriction of a hyperbolic isometry to U.

Definition: A *hyperbolic structure* on Σ is an atlas of coordinate charts on Σ such that

- The image of every coordinate chart is a disklike subset of $\boldsymbol{H}^2$.
- The overlap functions are local hyperbolic isometries.
- The atlas is maximal.

There are several auxilliary definitions which go along with this. A hyperbolic surface is *complete* if every Cauchy sequence converges. A hyperbolic surface is *compact* if every covering by open subsets has a finite subcovering. The notions of completeness and compactness are defined in any first year book on Real Analysis. All the examples constructed in this handout are compact.

2 The Associated Riemannian Structure

Here we show that a hyperbolic surface, as defined above, can automatically be considered as a smooth Riemannian surface.

Exercise 1: Prove that a local hyperbolic isometry is a smooth map.

In light of Exercise 1, a hyperbolic structure automatically gives an atlas of smooth coordinate charts. This atlas is not maximal, but then we can complete it to a maximal atlas using Zorn's lemma. Thus, every hyperbolic surface automatically is a smooth surface.

We can define a Riemannian metric on a hyperbolic surface Σ as follows. Let $p \in \Sigma$ be a point. Let (U, f) be a coordinate chart about p. This means that U is an open neighborhood of p and $f : U \to \boldsymbol{H}^2$ is a homeomorphism onto a disklike set. Let $V, W \in T_p(\Sigma)$ be two tangent vectors. This is to say $V = [\alpha]$ and $W = [\beta]$ where $\alpha, \beta : (-\epsilon, \epsilon) \to \Sigma$ are smooth curves with $\alpha(0) = \beta(0) = p$. We define

$$H_p(V, W) = G_{f(p)}((f \circ \alpha)'(0), (f \circ \beta)'(0)).$$

Here G is the Riemannian metric on the hyperbolic plane. In other words, we have just used the coordinate chart to transfer the metric on $\boldsymbol{H}^2$ to the tangent space $T_p\Sigma$ of Σ at p.

The fact that the overlap functions are all hyperbolic isometries implies that the above definition of the metric is independent of which coordinate chart is used.

3 Gluing Recipes

We would like a way to build lots of hyperbolic surfaces. Recall that a *convex geodesic polygon* is a convex subset of $\boldsymbol{H}^2$ whose boundary consists of a simple closed path of geodesic segments. I discussed these objects in detail at the end of Handout 7. Again, the convexity condition means that any two points in the polygon can be joined by a convex set which is contained in the set.

Let P be a geodesic polygon. Let $e \in P$ be an edge. Say a *decoration* of e is a labelling of e by both a number and an arrow. Say that a *decoration* of P is a decoration of every edge of P. Whenever we have built surfaces by gluing the sides of a polygon together−e.g. in the Asteroids example−we are always basing the construction on a decoration of the polygon.

We say that a *gluing recipe* for a hyperbolic surface is a finite list $P_1, ..., P_n$ of decorated polygons. There are some conditions we want to force:

- If some number appears as a label, then it appears as the label for exactly two edges. This condition guarantees that we will glue the edges together in pairs.

- If two edges have the same numerical label, then they have the same hyperbolic length. This allows us to make our gluing using (the restriction of) a hyperbolic isometry.

- Any *complete circuit* of angles adds up to 2π. This condition guarantees that a neighborhood of each vertex is locally isometric to $\boldsymbol{H}^2$.

The third condition requires some explanation. A *complete circuit* is a collection of edges

$$e_1, e_1', e_2, e_2', e_3, e_3', ..., e_k', e_1.$$

with the property, for all j, that e_j and e_j' have the same numerical label and e_j' and e_{j+1} are consecutive edges of the same polygon. (Here we are taking the indices cyclically, so that $k+1$ is set equal to 1.)

There is one subtle condition which we need also to require. Let v_j be the vertex incident to e_j' and e_{j+1}. Then the arrow along e_{j+1} points to v_j iff the arrow along e_{j+1}' points to v_{j+1}'. If you draw a few examples you will see the point of this last requirement. The point is that we want the edges in our chain to emanate from a single vertex in the quotient space.

The edges e_j and e_{j+1}' subtend an angle α_j and we want $\alpha_1 + ... + \alpha_k = 2\pi$.

4 The Basic Result

Here we will sketch the proof of

Theorem 4.1 *Any gluing recipe gives rise to a hyperbolic surface in a natural way.*

The proof comes in 4 steps.

4.1 Step 1

Given a gluing recipe we can form a surface Σ as follows: First of all, we start out with the metric space X which is the disjoint union of $P_1, ..., P_n$. We can do this by declaring $d(p,q) = 1$ if $p \in P_i$ and $q \in P_j$ with $j \neq i$. For $p, q \in P_j$ (the same polygon) we just the hyperbolic metric. So, you should picture X approximately as a stack of polygons hovering in the air.

Now we define an equivalence relation on X using the rule that $p \sim p'$ iff p and p' are corresponding points on like numbered edges. Here *corresponding* should be pretty obvious. Suppose e and e' are two like numbered edges, both having length λ. Then there is some t such that p is t units along e measured in the direction of the arrow. Likewise there is some t' such that p' is t' units along e'. Then p and p' are corresponding points iff $t = t'$.

The surface is defined as $\Sigma = X/\sim$.

4.2 Step 2

We would like to show that Σ is indeed a surface, so we have to construct an atlas of coordinate charts. Here is a general recipe for doing this. First of all, we're going to define some sets on a single polygon. We get our coordinate charts by suitably piecing these sets together.

Choose some very small ϵ_1 and let $U(P) \subset P$ be the set of points which are at least ϵ_1 from the boundary of P.

Exercise 2: Show that $U(P)$ is a disklike set if ϵ_1 is chosen small enough. Hint: Try to show that there is a point $p \in P$ such that the radial geodesics emanating from p only intersect the boundary of U in a single point. Then use the fairly obvious "polar coordinate map" to the disk.

Choose some small ϵ_2 and let e be an edge of P. Let $U(e)$ be the set of points $x \in P$ such that

- x is within ϵ_2 of e.
- x is at least ϵ from any vertex of P.

Finally, choose some ϵ_3. For any vertex v of P let $U(v)$ denote the set of points in P which are within ϵ_3 of v.

Exercise 3: Prove that the constants $\epsilon_1, \epsilon_2, \epsilon_3$ can be chosen so small that

- For any edge e the set $U(e)$ only intersects $U(P)$ and $U(v_1)$ and $U(v_2)$, where v_1 and v_2 are the two vertices incident to e.
- For any vertex v the set $U(v)$ only intersects $U(P)$ and $U(e_1)$ and $U(e_2)$, where e_1 and e_2 are the edges incident to v.

In fact, show that things work out if ϵ is sufficiently small and $\epsilon_j = \epsilon^j$ for $j = 1, 2, 3$. I mean the powers of a single constant. (Hint: draw a good schematic picture of all this.)

From now on we choose our constants to be $\epsilon^1, \epsilon^2, \epsilon^3$. If we have a finite list of polygons we fix a single ϵ which works for all of them (in the sense of Exercise 3) and then we make the above constructions.

4.3 Step 3

This step breaks into 3 substeps.

4.3.1 Step 3A: Near the middle

We want to define coordinate charts from Σ into $\boldsymbol{H}^2$. First, we can use the identity map on each set $U(P_j)$. This makes sense because each P_j is a subset of $\boldsymbol{H}^2$. Clearly these coordinate charts are homeomorphisms from open subsets of Σ to disklike sets.

4.3.2 Step 3B: Near the Edges

If e and e' are two edges which have the same numerical label, then we can define $h : U(e) \to \boldsymbol{H}^2$ and $h(U(e') \to \boldsymbol{H}^2$ in such a way that

- The map h is the identity map composed with a hyperbolic isometry.
- The map h' is the identity map composed with a hyperbolic isometry.
- $h(e \cap U(e)) = h'(e' \cap U(e'))$ and the arrows go the right way.
- $h(U(e))$ and $h'(U(e')$ lie on opposite sides of $H(e) = H(e')$.

This is pretty obvious. We can first define h and h' as the identity maps, and then compose one of our maps with a suitable isometry to adjust things. The main point here is that $U(e) \cap e$ and $U(e') \cap e'$ are open geodesic segments of the same length.

Exercise 4: Draw a picture of $h(U(e)) \cup h'(U'(e))$. Show that this set a disklike set if ϵ is chosen small enough.

Note that $U(e)$ and $U(e')$ piece together on Σ to give an open neighborhood of any point of the form $[x]$ where $x \in U(e) \cap e$ or $x \in U(e') \cap e'$. It is not hard to show that h and h' together define a homeomorphism from $(U(e) \cap U(e')/\sim$ onto $h(U(e)) \cup h'(U(e'))$.

4.3.3 Step 3C: Near the Vertices

Let B denote the open ball of radius ϵ^3 about, the point $i = \sqrt{-1} \in \boldsymbol{H}^2$. Let $[v]$ be any vertex of Σ, and let $v_1, ..., v_k$ be the complete list of the members of the equivalence class of v. The sets $U(v_1), ..., U(v_k)$ are little “pizza slices”. All that remains is to analyze how these pizza slices fit together.

Exercise 5: Prove that we can find isometric maps $h_i : U(v_i) \to \boldsymbol{H}^2$ such that

- $h_j(v_j) = j$ for all j.
- $h_j(U(v_j)) \cap h_{j+1}(U(v_{j+1}))$ is a half-open geodesic segment emanating from $\sqrt{-1}$.

Under these circumstances, prove that $\bigcup h_j(U(v_j)) = B$. (Hint: Picture these open sets as fitting around i like the slices of pizza. The 2π condition makes things work.

The pieces $U(v_1), ..., U(v_k)$ fit together to make a neighborhood of v in Σ and it is not hard to show that the maps $h_1, ..., h_k$ piece together to give a homeomorphism from

$$U(v_1) \cup ... \cup U(v_k)/\sim$$

to B.

4.4 Step 4

From the way we have defined things, the overlap functions are all local hyperbolic isometries, so we have found an atlas on Σ whose overlap functions are local hyperbolic isometries. We can complete this to a maximal atlas, if we like, using Zorn's lemma.

5 Some Examples

Exercise 6: Prove that there is a regular convex $4n$ gon, with angles $\pi/2n$, provided that $n \geq 2$. Call this polygon P_{4n}. Decorate P_{4n} by giving the opposite sides and making the arrows point in the same direction. Prove that P_{4n}, as decorated, is a gluing diagram for a hyperbolic surface.

Exercise 7: Prove that there exists a right angled regular hexagon. Construct a decoration of $2n$ such hexagons in such a way that it is the gluing diagram for a hyperbolic surface. Here $n \geq 2$ should be even.

exercise 8: (Challenge) If you take $n = 2$ in Exercises 6 and 7 you get homeomorphic surfaces. Prove that they are not isometric.

6 A Glimpse at Moduli Space

We can fix a homeomorphism type of surface, for example the surface of genus $g = 2$. Let M_g denote the set of all different hyperbolic surfaces which are homeomorphic to our fixed surface. Two such surfaces are considered "the same" if there is an isometry between them.

Exercise 9 (Challenge): Prove that M_g has uncountably many points for any given genus g. Hint: Modify the construction in Exercise 7, using hexagons which have 3-fold rather than 6-fold symmetry.

We would like to make M_g into a metric space! Let Σ_1 and Σ_2 be two hyperbolic surfaces. Let $d(\Sigma_1, \Sigma_2)$ be $\log(K)$, where K is the infimal number such that there is a homeomorphism $h : \Sigma_1 \to \Sigma_2$ which is K-bilipschitz. This means that

$$\frac{1}{K} d(x,y) \leq d(h(x), h(y)) \leq K d(x,y); \qquad \forall x, y \in \Sigma_1.$$

Exercise 10: Prove that M_g is a metric space, when it is equipped with d. The only hard part of this exercise is showing that $d > 0$ when Σ_1 and Σ_2 are not isometric.

In the past 50 years there has been intense interest in the spaces M_g, when it is given various metrics and auxilliary structures. These notes are just meant to give you a little glimpse of it.