# Math 181 Handout 8

Rich Schwartz

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The purpose of this handout is to define hyperbolic surfaces and to present a general method of constructing them out of convex geodesic polygons.

## 1 Definition

Let $\Sigma$ be a surface. This to say that $\Sigma$ is a metric space, and every point of $\Sigma$ has a neighborhood which is homeomorphic to $\boldsymbol{R}^2$. Actually, it is a little bit annoying to always have to map the neighborhoods of $\Sigma$ to all of $\boldsymbol{R}^2$. So, we define a *disklike subset* of $\boldsymbol{R}^2$ to be a subset which is homeomorphic to $\boldsymbol{R}^2$. For instance, an open disk is disklike. So is an open (solid) square. So, we can equally well say that a surface is a metric space such that every point has a neighborhood which is homeomorphic to a disklike subset of $\boldsymbol{R}^2$. If a disklike set is contained in the upper half plane, we call it a disklike subset of $\boldsymbol{H}^2$.

Let $U$ and $V$ be two open subset of $\boldsymbol{H}^2$. Say that a map $f : U \to V$ is a *local hyperbolic isometry* if the restriction of $f$ to each open component of $U$ agrees with the restriction of a hyperbolic isometry. The easiest case to think about is when $U$ and $V$ are both connected. Then $f : U \to V$ is a local isometry iff $f$ is the restriction of a hyperbolic isometry to $U$.

**Definition:** A *hyperbolic structure* on $\Sigma$ is an atlas of coordinate charts on $\Sigma$ such that

- The image of every coordinate chart is a disklike subset of $\boldsymbol{H}^2$.
- The overlap functions are local hyperbolic isometries.
- The atlas is maximal.

There are several auxilliary definitions which go along with this. A hyperbolic surface is *complete* if every Cauchy sequence converges. A hyperbolic surface is *compact* if every covering by open subsets has a finite subcovering. The notions of completeness and compactness are defined in any first year book on Real Analysis. All the examples constructed in this handout are compact.

# 2 The Associated Riemannian Structure

Here we show that a hyperbolic surface, as defined above, can automatically be considered as a smooth Riemannian surface.

**Exercise 1:** Prove that a local hyperbolic isometry is a smooth map.

In light of Exercise 1, a hyperbolic structure automatically gives an atlas of smooth coordinate charts. This atlas is not maximal, but then we can complete it to a maximal atlas using Zorn's lemma. Thus, every hyperbolic surface automatically is a smooth surface.

We can define a Riemannian metric on a hyperbolic surface $\Sigma$ as follows. Let $p \in \Sigma$ be a point. Let $(U, f)$ be a coordinate chart about $p$. This means that $U$ is an open neighborhood of $p$ and $f : U \to \boldsymbol{H}^2$ is a homeomorphism onto a disklike set. Let $V, W \in T_p(\Sigma)$ be two tangent vectors. This is to say $V = [\alpha]$ and $W = [\beta]$ where $\alpha, \beta : (-\epsilon, \epsilon) \to \Sigma$ are smooth curves with $\alpha(0) = \beta(0) = p$. We define

$$H_p(V, W) = G_{f(p)}((f \circ \alpha)'(0), (f \circ \beta)'(0)).$$

Here $G$ is the Riemannian metric on the hyperbolic plane. In other words, we have just used the coordinate chart to transfer the metric on $\boldsymbol{H}^2$ to the tangent space $T_p\Sigma$ of $\Sigma$ at $p$.

The fact that the overlap functions are all hyperbolic isometries implies that the above definition of the metric is independent of which coordinate chart is used.

# 3 Gluing Recipes

We would like a way to build lots of hyperbolic surfaces. Recall that a *convex geodesic polygon* is a convex subset of $\boldsymbol{H}^2$ whose boundary consists of a simple closed path of geodesic segments. I discussed these objects in detail at the end of Handout 7. Again, the convexity condition means that any two points in the polygon can be joined by a convex set which is contained in the set.

Let $P$ be a geodesic polygon. Let $e \in P$ be an edge. Say a *decoration* of $e$ is a labelling of $e$ by both a number and an arrow. Say that a *decoration* of $P$ is a decoration of every edge of $P$. Whenever we have built surfaces by gluing the sides of a polygon together−e.g. in the Asteroids example−we are always basing the construction on a decoration of the polygon.

We say that a *gluing recipe* for a hyperbolic surface is a finite list $P_1, ..., P_n$ of decorated polygons. There are some conditions we want to force:

- If some number appears as a label, then it appears as the label for exactly two edges. This condition guarantees that we will glue the edges together in pairs.
- If two edges have the same numerical label, then they have the same hyperbolic length. This allows us to make our gluing using (the restriction of) a hyperbolic isometry.
- Any *complete circuit* of angles adds up to $2\pi$. This condition guarantees that a neighborhood of each vertex is locally isometric to $\boldsymbol{H}^2$.

The third condition requires some explanation. A *complete circuit* is a collection of edges

$$e_1, e'_1, e_2, e'_2, e_3, e'_3, ..., e'_k, e_1.$$

with the property, for all $j$, that $e_j$ and $e'_j$ have the same numerical label and $e'_j$ and $e_{j+1}$ are consecutive edges of the same polygon. (Here we are taking the indices cyclically, so that $k+1$ is set equal to 1.)

There is one subtle condition which we need also to require. Let $v_j$ be the vertex incident to $e'_j$ and $e_{j+1}$. Then the arrow along $e_{j+1}$ points to $v_j$ iff the arrow along $e'_{j+1}$ points to $v'_{j+1}$. If you draw a few examples you will see the point of this last requirement. The point is that we want the edges in our chain to emanate from a single vertex in the quotient space.

The edges $e_j$ and $e'_{j+1}$ subtend an angle $\alpha_j$ and we want $\alpha_1 + ... + \alpha_k = 2\pi$.

# 4 The Basic Result

Here we will sketch the proof of

**Theorem 4.1** *Any gluing recipe gives rise to a hyperbolic surface in a natural way.*

The proof comes in 4 steps.

## 4.1 Step 1

Given a gluing recipe we can form a surface $\Sigma$ as follows: First of all, we start out with the metric space $X$ which is the disjoint union of $P_1, ..., P_n$. We can do this by declaring $d(p, q) = 1$ if $p \in P_i$ and $q \in P_j$ with $j \neq i$. For $p, q \in P_j$ (the same polygon) we just the hyperbolic metric. So, you should picture $X$ approximately as a stack of polygons hovering in the air.

Now we define an equivalence relation on $X$ using the rule that $p \sim p'$ iff $p$ and $p'$ are corresponding points on like numbered edges. Here *corresponding* should be pretty obvious. Suppose $e$ and $e'$ are two like numbered edges, both having length $\lambda$. Then there is some $t$ such that $p$ is $t$ units along $e$ measured in the direction of the arrow. Likewise there is some $t'$ such that $p'$ is $t'$ units along $e'$. Then $p$ and $p'$ are corresponding points iff $t = t'$.

The surface is defined as $\Sigma = X/\sim$.

## 4.2 Step 2

We would like to show that $\Sigma$ is indeed a surface, so we have to construct an atlas of coordinate charts. Here is a general recipe for doing this. First of all, we're going to define some sets on a single polygon. We get our coordinate charts by suitably piecing these sets together.

Choose some very small $\epsilon_1$ and let $U(P) \subset P$ be the set of points which are at least $\epsilon_1$ from the boundary of $P$.

**Exercise 2:** Show that $U(P)$ is a disklike set if $\epsilon_1$ is chosen small enough. Hint: Try to show that there is a point $p \in P$ such that the radial geodesics emanating from $p$ only intersect the boundary of $U$ in a single point. Then use the fairly obvious "polar coordinate map" to the disk.

Choose some small $\epsilon_2$ and let $e$ be an edge of $P$. Let $U(e)$ be the set of points $x \in P$ such that

- $x$ is within $\epsilon_2$ of $e$.
- $x$ is at least $\epsilon$ from any vertex of $P$.

Finally, choose some $\epsilon_3$. For any vertex $v$ of $P$ let $U(v)$ denote the set of points in $P$ which are within $\epsilon_3$ of $v$.

**Exercise 3:** Prove that the constants $\epsilon_1, \epsilon_2, \epsilon_3$ can be chosen so small that

- For any edge $e$ the set $U(e)$ only intersects $U(P)$ and $U(v_1)$ and $U(v_2)$, where $v_1$ and $v_2$ are the two vertices incident to $e$.
- For any vertex $v$ the set $U(v)$ only intersects $U(P)$ and $U(e_1)$ and $U(e_2)$, where $e_1$ and $e_2$ are the edges incident to $v$.

In fact, show that things work out if $\epsilon$ is sufficiently small and $\epsilon_j = \epsilon^j$ for $j = 1, 2, 3$. I mean the powers of a single constant. (Hint: draw a good schematic picture of all this.)

From now on we choose our constants to be $\epsilon^1, \epsilon^2, \epsilon^3$. If we have a finite list of polygons we fix a single $\epsilon$ which works for all of them (in the sense of Exercise 3) and then we make the above constructions.

## 4.3 Step 3

This step breaks into 3 substeps.

### 4.3.1 Step 3A: Near the middle

We want to define coordinate charts from $\Sigma$ into $\boldsymbol{H}^2$. First, we can use the identity map on each set $U(P_j)$. This makes sense because each $P_j$ is a subset of $\boldsymbol{H}^2$. Clearly these coordinate charts are homeomorphisms from open subsets of $\Sigma$ to disklike sets.

### 4.3.2 Step 3B: Near the Edges

If $e$ and $e'$ are two edges which have the same numerical label, then we can define $h : U(e) \to \boldsymbol{H}^2$ and $h(U(e') \to \boldsymbol{H}^2$ in such a way that

- The map $h$ is the identity map composed with a hyperbolic isometry.
- The map $h'$ is the identity map composed with a hyperbolic isometry.
- $h(e \cap U(e)) = h'(e' \cap U(e'))$ and the arrows go the right way.
- $h(U(e))$ and $h'(U(e')$ lie on opposite sides of $H(e) = H(e')$.

This is pretty obvious. We can first define $h$ and $h'$ as the identity maps, and then compose one of our maps with a suitable isometry to adjust things. The main point here is that $U(e) \cap e$ and $U(e') \cap e'$ are open geodesic segments of the same length.

**Exercise 4:** Draw a picture of $h(U(e)) \cup h'(U'(e))$. Show that this set a disklike set if $\epsilon$ is chosen small enough.

Note that $U(e)$ and $U(e')$ piece together on $\Sigma$ to give an open neighborhood of any point of the form $[x]$ where $x \in U(e) \cap e$ or $x \in U(e') \cap e'$. It is not hard to show that $h$ and $h'$ together define a homeomorphism from $(U(e) \cap U(e')/\sim$ onto $h(U(e)) \cup h'(U(e'))$.

### 4.3.3 Step 3C: Near the Vertices

Let $B$ denote the open ball of radius $\epsilon^3$ about, the point $i = \sqrt{-1} \in \boldsymbol{H}^2$. Let $[v]$ be any vertex of $\Sigma$, and let $v_1, ..., v_k$ be the complete list of the members of the equivalence class of $v$. The sets $U(v_1), ..., U(v_k)$ are little "pizza slices". All that remains is to analyze how these pizza slices fit together.

**Exercise 5:** Prove that we can find isometric maps $h_i : U(v_i) \to \boldsymbol{H}^2$ such that

- $h_j(v_j) = j$ for all $j$.
- $h_j(U(v_j)) \cap h_{j+1}(U(v_{j+1}))$ is a half-open geodesic segment emanating from $\sqrt{-1}$.

Under these circumstances, prove that $\bigcup h_j(U(v_j)) = B$. (Hint: Picture these open sets as fitting around $i$ like the slices of pizza. The $2\pi$ condition makes things work.

The pieces $U(v_1), ..., U(v_k)$ fit together to make a neighborhood of $v$ in $\Sigma$ and it is not hard to show that the maps $h_1, ..., h_k$ piece together to give a homeomorphism from

$$U(v_1) \cup ... \cup U(v_k)/\sim$$

to $B$.

### 4.4 Step 4

From the way we have defined things, the overlap functions are all local hyperbolic isometries, so we have found an atlas on $\Sigma$ whose overlap functions are local hyperbolic isometries. We can complete this to a maximal atlas, if we like, using Zorn's lemma.

## 5 Some Examples

**Exercise 6:** Prove that there is a regular convex $4n$ gon, with angles $\pi/2n$, provided that $n \geq 2$. Call this polygon $P_{4n}$. Decorate $P_{4n}$ by giving the opposite sides and making the arrows point in the same direction. Prove that $P_{4n}$, as decorated, is a gluing diagram for a hyperbolic surface.

**Exercise 7:** Prove that there exists a right angled regular hexagon. Construct a decoration of $2n$ such hexagons in such a way that it is the gluing diagram for a hyperbolic surface. Here $n \geq 2$ should be even.

**exercise 8: (Challenge)** If you take $n = 2$ in Exercises 6 and 7 you get homeomorphic surfaces. Prove that they are not isometric.

## 6 A Glimpse at Moduli Space

We can fix a homeomorphism type of surface, for example the surface of genus $g = 2$. Let $M_g$ denote the set of all different hyperbolic surfaces which are homeomorphic to our fixed surface. Two such surfaces are considered "the same" if there is an isometry between them.

**Exercise 9 (Challenge):** Prove that $M_g$ has uncountably many points for any given genus $g$. Hint: Modify the construction in Exercise 7, using hexagons which have 3-fold rather than 6-fold symmetry.

We would like to make $M_g$ into a metric space! Let $\Sigma_1$ and $\Sigma_2$ be two hyperbolic surfaces. Let $d(\Sigma_1, \Sigma_2)$ be $\log(K)$, where $K$ is the infimal number such that there is a homeomorphism $h : \Sigma_1 \to \Sigma_2$ which is $K$-bilipschitz. This means that

$$\frac{1}{K} d(x,y) \leq d(h(x), h(y)) \leq K d(x,y); \qquad \forall x, y \in \Sigma_1.$$

**Exercise 10:** Prove that $M_g$ is a metric space, when it is equipped with $d$. The only hard part of this exercise is showing that $d > 0$ when $\Sigma_1$ and $\Sigma_2$ are not isometric.

In the past 50 years there has been intense interest in the spaces $M_g$, when it is given various metrics and auxilliary structures. These notes are just meant to give you a little glimpse of it.