Math 20 Final 13 May 2008

Instructions: Show all your work. This is a closed book exam. You cannot use calculators. Each problem is worth 20 points.

1. Find the equation for the plane containing the line $(1+2t, 2+3t, 4-t)$ and the point $(-1, 3, 2)$.

2. Consider the function $f(x, y) = \sin(2\pi x)\cos(2\pi y)$, defined in the plane. Find all the local maxima, local minima, and saddle points for f.

3. Suppose that $x + y + z = 9$, and all numbers are positive. Use Lagrange multipliers to find the maximum value of xyz.

4. Let T be the triangle whose vertices are $(1, 0, 0)$ and $(0, 2, 0)$ and $(0, 0, 3)$. Let $\delta(x, y, z) = x$ be the local density of T. (This function is defined on all of space, but you can just restrict it to T.) Find the distance from the center of mass of T to the xy plane. Assume that T has mass density δ.

5. In this problem, you just have to set up the integrals. You don't have to evaluate them.

- Let R be the solid region inside the cylinder $x^2+y^2 = 1$, above the plane $z = 0$, and below the plane $x + y + z = 10$. Write down an integral, in Cartesian coordinates, that computes the moment of inertia of R about the z axis.
- Let S_1 be the points (x, y, z) satisfying $x^2 + y^2 + z^2 \leq 4$ and $x \geq 0$. Here S_1 is a solid half-ball. Let S_2 be the cylinder consisting of all points that are within 1 unit of the line $(1, 0, t)$. So, S_2 is a vertical cylinder. Let S denote the set of points that are in S_1 but not in S_2. Write down the integrals, in cylindrical coordinates, that compute the center of mass of S.

6. Compute the line integral

$$\int_C x\ dy - y\ dx$$

where C is the portion of the parabola $y = x^2$ connecting $(-1, 1)$ to $(1, 1)$ and oriented from the first point to the second one.

7. Consider the parametric surface given by $r(u, v) = (uv, u + v, u^2 - v^2)$. Let T be the solid triangle in the (u, v)-plane having vertics $(1, 1)$, $(1, 2)$, and $(2, 2)$. Let $\overrightarrow{F}(x, y, z) = (x, y, z)$. Let $R = r(T)$. Compute

$$\pm \int_R \overrightarrow{F} \cdot \overrightarrow{n}\, dx.$$

The $(\pm)$ means that switching the normal direction on R will change the sign, but we don't care which choice you make.

8. Let C be the ellipse satisfying the equation $4x^2 + y^2 = 1$. We orient C counterclockwise. Let

$$\overrightarrow{F}(x, y) = (2y + e^x \cos(y), -e^x \sin(y) - 3x).$$

Using Green's theorem, compute the line integral

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds.$$

9. Let Q be the cube satisfying the equation $1 \leq x \leq 2$ and $2 \leq y \leq 3$ and $0 \leq z \leq 1$. One of the faces of Q lies in the (x, y) plane. Let S be the union of the other 5 faces of Q. We orient S with the normal pointing into Q. Consider the vector field

$$\overrightarrow{F}(x, y, z) = (2x - xz, 3y - yz, z^2).$$

Compute the flux of F through the surface S. Hint: the Divergence Theorem will help.

10. Consider the vector field $\overrightarrow{F}(x, y, z) = (-y, x, x)$. Let S be a surface whose boundary is the union of two circles C_1 and C_2. Also suppose that S stays above the (x, y) plane, except for its boundary. (This assumption forces C_1 and C_2 to both be oriented in the same way, either clockwise or counterclockwise.) Here C_1 is the circle of radius 1 in the (x, y) plane centered at the origin and C_2 is the circle in the (x, y) plane of radius 2 centered at $(10, 10)$. Basically, you should picture S like a piece of macaroni glued to the table at its ends. Use Stokes' Theorem to compute $|X|$, where X is the flux of $\text{curl}(\overrightarrow{F})$ through S. In other words, you only have to compute the flux up to sign.