Practice Final Solutions:

1. You can find two points on the line by setting $t = 0$ and $t = 1$. This gives $(1, 2, 4)$ and $(3, 5, 3)$. The third point is $(-1, 3, 2)$. Call these points P_1, P_2, P_3. Let $n = (P_3 - P_1) \times (P_2 - P_1) = (5, -6, -8)$. The equation for the plane is

$$((x, y, z) - P_1) \cdot n = 0.$$

This works out to $5x - 6y - 8z = -39$

2. You just have to find the max and min and saddle points for $0 \le x < 1$ and $0 \le y < 1$ because the points repeat with period 1 in both coordinates. That is $f(x+m, y+n) = f(x, y)$ when m and n are integers. Both $\sin(2\pi x)$ and $\cos(2\pi y)$ have values in the interval $[-1, 1]$. So, $f(x, y)$ ranges between -1 and 1. The max points occur when $f(x, y) = 1$. This forces one of two situations:

- $\sin(2\pi x) = 1$ and $\cos(2\pi y) = 1$
- $\sin(2\pi x) = -1$ and $\cos(2\pi y) = -1$.

In the first case, we get $(1/4, 0)$. In the second case, we get $(3/4, 1/2)$. The analysis for the min points is similar. The result: $(1/4, 1/2)$ and $(3/4, 0)$ are the min points. For the saddle points, set $\nabla f = 0$ and apply the second derivative test. The saddle points appear exactly midway between all the max points and the min points. They are $(0, 1/4)$ and $(0, 3/4)$ and $(1/2, 1/4)$ and $(1/2, 3/4)$.

3. Let $f = x + y + z$ and $g = xyz$. We have $\nabla f = (1, 1, 1)$ and $\nabla g = (yz, xz, xy)$. The Lagrange multiplier equation leads to

$$(xyz, xyz, xyz) = \lambda(x, y, z).$$

I multiplied the first coords by x, the second by y, and the third by z. This equation gives $x = y = z$. Hence $x = y = z = 3$. This means that $xyz = 27$ is the largest possible value.

4. (sketch) The centroid (or center of mass) is $(\overline{x}, \overline{y}, \overline{z})$. It doesn't matter where $\overline{x}$ and $\overline{y}$ are, because we're taking the distance to the xy plane. So, we just have to compute $\overline{z}$. We have the equation

$$\overline{z} = \frac{\int\int_T xzdS}{\int\int_T xdS}.$$

All the points on the triangle satisfy the equation $6x + 3y + 2z = 6$. So, you can parametrize T by the equation

$$S(x, y) = (x, y, (6 - 3y - 2x)/2).$$

Here x and y lie in the planar triangle Δ with vertices $(0, 0)$ and $(1, 0)$ and $(0, 2)$. We compute

$$dS = \left\| S_x \times S_y \right\| = \sqrt{17}/2 \; dxdy$$

The important thing is that this is just a constant factor. This factor appears in the top and bottom of the integrals involved, and therefore cancels. So,

$$\overline{z} = \frac{1}{2} \frac{\int\int_\Delta x(6 - 3y - 2x) \; dxdy}{\int\int_\Delta x \; dxdy}.$$

This is a straightforward integral, which I won't do.

5a

$$\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{10} (x^2 + y^2) \; dzdydx.$$

5b We want to find $(\overline{x}, \overline{y}, \overline{z})$. By symmetry $\overline{z} = 0$. For the other two quantities, the integrals involved all have the form: Consider the integral

$$I(A) = \int_0^\pi \int_{2\cos(\theta)}^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} Ardzd\theta dr.$$

Then

$$\overline{x} = I(r\cos(\theta))/I(1); \qquad \overline{y} = I(r\sin(\theta)/I(1).$$

6. Parametrize the curve by $r(x) = (x, x^2)$ with x going from $-1, 1$. Note that $r'(x) = (1, 2x)$. The line element is $xdy - ydx$. As a vector field it is $(-y, x)$. Along the curve it is $(-x^2, x)$. So, the integral is

$$\int_{-1}^{1} (-x^2, x) \cdot (1, 2x) dx = \int_{-1}^{1} x^2 \; dx = 2/3.$$

7. There is a typo in the problem. You're supposed to compute

$$\int\int_{r(T)} \overrightarrow{F} \cdot \overrightarrow{n}\ dS.$$

This is a straightforward problem about evaluating a surface integral. This is the same as

$$\int\int_{T} F(r(u,v)) \cdot N(u,v)\ dudv,$$

where

$$N = r_u \times r_v.$$

This is a messy integral, but it only involves polynomials in u and v.

8. In spite of the complicated formula, F has curl -5 at every point. So, by Green's theorem, we're just computing 5 times the area of the ellipse. The ellipse has half the area of the unit circle. So, the final answer is $5\pi/2$.

9. F is horizontal in the face contained in the xy plane, so it has no flux through this face. Therefore, the flux through the other 5 faces is the same as the flux through the surface of the whole cube. We compute $\div(F) = 5$ at every point. So, the answer is just 5 times the volume of the cube. Since the cube has volume 1, the final answer is 5.

10. By Stokes' Theorem, you just have to integrate $F = (-y, x, x)$ around the two circles in the plane. Since the plane is $z = 0$, you are just integrating the plane vector field $G = (P, Q) = (-y, x)$ around the two circles. By Green's Theorem, the line integral is just the integral of $Q_x - P_y = 2$ over the two disks. That is, you are computing twice the area of the two disks. This comes out to 10π.