Notes on Green's Theorem

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In class, I botched one of the calculations for the proof of Green's theorem, getting the sign wrong several times before finally getting it right. In these notes, I'll re-do this calculation, and also the other ones connected to it.

Here is the setup:

- R is the triangular region with vertices $(0,0)$, $(h,0)$ and $(0,k)$.
- $F=(P,Q)$ is a vector field defined throughout R.
- C is the boundary of R, oriented counterclockwise.
- We want to show that

$$(1) \qquad \iint_R (Q_x - P_y)dA = \oint_C P\ dx + Q\ dy.$$

- Equation (1) breaks up into 2 pieces:

$$(2) \qquad \iint_R Q_x\ dA = \oint_C Q\ dy$$

  and

$$(3) \qquad -\iint_R P_y\ dA = \oint_C P\ dx.$$

  I'll work out Equation (2). The calculation for Equation (3) is similar, except that you switch the roles played by the variables x and y.

The Left Hand Side: Let's look at the left hand side of Equation (2). The hypotenuse of R lies on the line

$$\frac{x}{h}+\frac{y}{k}=1.$$

In other words,

$$x = g(y) := h(1 - (y/k)).$$

Therefore, the left hand side of Equation 2 is

$$\int_0^k \int_{x=0}^{g(y)} \frac{\partial Q}{\partial x}(x, y)\ dx\ dy.$$

By the fundamental theorem of calculus,

$$\int_{x=0}^{g(y)} \frac{\partial Q}{\partial x}(x, y)\ dx = Q(g(y), 0) - Q(0, y).$$

Therefore, the left hand side of (2) equals

$$(4) \qquad \int_0^k Q(g(y), y)\ dy - \int_0^k Q(0, y)\ dy.$$

The Right Hand Side: Now let's look at the right hand side of (2). The curve C is the union of 3 pieces:

1. The line segment C_1 connecting $(0, 0)$ to $(h, 0)$.
2. The line segment C_2 connecting $(0, k)$ to $(0, 0)$.
3. The line segment C_3 connecting $(h, 0)$ to $(0, k)$.

Accordingly, the right hand side of (2) breaks up into 3 integrals. We have

$$\int_{C_1} Q\ dy = 0,$$

because $dy(v) = 0$ for any horizontal vector v, and C_1 is a horizontal segment. The second integral is the one I messed up in class. Here is an easier way to do it, and get the sign right. Let D_2 denote the line segment connecting $(0, 0)$ to $(0, k)$. Then C_2 and D_2 are the same segment but oriented in opposite directions. From a basic property of line integrals, we have

$$\int_{C_2} Q\ dy = -\int_{D_2} Q\ dy.$$

We can parametrize D_2 using the curve $r(y) = (0, y)$, from $y = 0$ to $y = k$. But then

$$\int_{D_2} Q\ dy = \int_0^k Q(0, y)\ dy.$$

Hence

$$\int_{C_2} Q \; dy = -\int_0^k Q(0,y) \; dy.$$

This is the last part of Equation (4).

The curve

$$r(y) = (g(y), y)$$

parametrizes C_3 for $y = 0$ to k. Therefore

$$\int_{C_3} Q \; dy = \int_0^k Q(g(y), y) \; dy.$$

This is the first part of Equation (4). Comparing our last two integrals with Equation (4), we see that the left hand side of (2) equals the right hand side. This is what we wanted.