Math 20 Midterm 1 solutions

1. Compute $\nabla f = (2xz + 4z^2, -6y, x^2 + 8xz)$. Plug in $(1, 1, 1)$ to get $N := \nabla f(1, 1, 1) = (6, -6, 9)$. The equation is $(x, y, z) \cdot N = (1, 1, 1).N = 9$. This works out to $6x - 6y + 9z = 9$, or $2x - 2y + 3z = 3$.

2. The gradient works out to $N := \nabla f(p) = (5, 2, 2)$. The formula for the directional derivative is $D_v f(p) = v \cdot (5, 2, 2)$.

a. Taking $v = (1, 0, 0)$ is the easiest solution.

b. The desired vector is perpendicular to both $N$ and $w = (0, -1, 1)$. Taking $N \times w$ gives $(4, -5, -5)$. Divide by the norm to get a unit vector in the same direction. The result is: $v = (4, -5, -5)/\sqrt{66}$.

c. Note that $\|N\| = \sqrt{33} < 6$. But $|D_v f(p)| = \|N\| \cos(\theta) < 6$.

Here $\theta$ is the angle between $v$ and $N$.

d. Since $\|N\| > 4$ there is, by the intermediate value theorem, some choice of $\theta$ such that $\|N\| \cos(\theta) = 4$. But then any unit vector $v$ making an angle $\theta$ with $N$ has the desired property. Drawing a picture in space, you can see that all such unit vectors lie on a cone, and there infinitely many unit vectors on a cone.

3. First, the critical points: The derivatives are defined everywhere. So, we just want to find the places where $\nabla f = 0$. Solving $\nabla f(x, y) = (y + 1, x) = (0, 0)$ gives $(x, y) = (0, -1)$. This is the only critical point. At this point,

$A = f_{xx}(0, -1) = 0; \quad B = f_{xy}(0, -1) = 1; \quad C = f_{yy}(0, -1) = 0; \quad \Delta = AC - B^2 = -1.$

The second derivative test says that this one critical point is a saddle.

Now for the max and min. Since there is only one critical point in $D$, and this point is a saddle, the max and min occur on the boundary. Use
Lagrange multipliers, for the constraint function \( g(x, y) = x^2 + y^2 - 3 \). The basic equation is

\[
(y + 1, x) = \lambda(2x, 2y).
\]

First let’s deal with exceptional cases. If \( x = 0 \) then \( \lambda = 0 \). But then \( y = -1 \). This isn’t a boundary point. If \( y = 0 \) then \( x = 0 \) no matter what value \( \lambda \) takes. The point \((0,0)\) is not a boundary point either. So, we have \( x \) and \( y \) both nonzero. This means that we can write the above equation as

\[
\frac{y + 1}{x} = \frac{x}{y}.
\]

Therefore \( x^2 = y^2 + y \). Plugging this into the equation \( g(x, y) = 0 \) we get

\[
2y^2 + y - 3 = 0.
\]

The two solutions are \( y = 1 \) and \( y = -3/2 \). When \( y = 1 \), we have \( x = \pm \sqrt{2} \). When \( y = -3/2 \), we have \( x = \pm 3/\sqrt{2} \). So, the candidate points are

\[
(1, \sqrt{2}); \quad (1, -\sqrt{2}); \quad (-3/2, \sqrt{3}/2); \quad (-3/2, -\sqrt{3}/2).
\]

Evaluating \( f \) at these points, we get:

\[
1 + \sqrt{2}; \quad 1 - \sqrt{2}; \quad -3/2 - 3\sqrt{3}/4; \quad -3/2 + 3\sqrt{3}/4.
\]

The max is \( \sqrt{2} + 1 \) and the min is \( -3/2 - 3\sqrt{3}/4 \).

4: Write \( r_1(t) = (a(t), 0) \) and \( r_2(t) = (b(t), c(t)) \). The square distance is given by

\[
f(a, b, c) = (b - a)^2 + c^2.
\]

Then

\[
E(t) = f(a(t), b(t), c(t)).
\]

The chain rule gives

\[
\frac{dE}{dt}(0) = \nabla f(1, 2, 3) \cdot (a'(0), b'(0), c'(0)).
\]

Here \( a' = da/dt \), etc. We have

\[
\nabla f(1, 2, 3) = \left[(2(a - b), 2(b - a), 2c)\right]_{(1,2,3)} = (-2, 2, 6),
\]

We have \( a'(0) = 2 \) and \( b'(0) = c'(0) = 1 \). The final answer is given by the dot product \((-2, 2, 6) \cdot (2, 1, 1) = 4\).