Math 20 Midterm 1 solutions

1. Compute $\nabla f = (2xz + 4z^2, -6y, x^2 + 8xz)$. Plug in $(1,1,1)$ to get $N := \nabla f(1,1,1) = (6,-6,9)$. The equation is $(x,y,z) \cdot N = (1,1,1).N = 9$. This works out to $6x - 6y + 9z = 9$, or $2x - 2y + 3z = 3$.

2. The gradient works out to $N := \nabla f(p) = (5,2,2)$. The formula for the directional derivative is

$$D_v f(p) = v \cdot (5,2,2).$$

a. Taking $v = (1,0,0)$ is the easiest solution.
b. The desired vector is perpendicular to both N and $w = (0,-1,1)$. Taking $N \times w$ gives $(4,-5,-5)$. Divide by the norm to get a unit vector in the same direction. The result is: $v = (4,-5,-5)/\sqrt{66}$.
c. Note that $\|N\| = \sqrt{33} < 6$. But

$$|D_v f(p)| = \|N\| |\cos(\theta)| < 6.$$

Here θ is the angle between v and N.
d. Since $\|N\| > 4$ there is, by the intermediate value theorem, some choice of θ such that $\|N\|\cos(\theta) = 4$. But then any unit vector v making an angle θ with N has the desired property. Drawing a picture in space, you can see that all such unit vectors lie on a cone, and there infinitely many unit vectors on a cone.

3. First, the critical points: The derivatives are defined everywhere. So, we just want to find the places where $\nabla f = 0$. Solving

$$\nabla f(x,y) = (y+1, x) = (0,0)$$

gives $(x,y) = (0,-1)$. This is the only critical point. At this point,

$$A = f_{xx}(0,-1) = 0; \qquad B = f_{xy}(0,-1) = 1; \qquad C - f_{yy}(0,-1) = 0;$$

$$\Delta = AC - B^2 = -1.$$

The second derivative test says that this one critical point is a saddle.

Now for the max and min. Since there is only one critical point in D, and this point is a saddle, the max and min occur on the boundary. Use

Lagrange multipliers, for the constraint function $g(x, y) = x^2 + y^2 - 3$. The basic equation is

$$(y + 1, x) = \lambda(2x, 2y).$$

First let's deal with exceptional cases. If $x = 0$ then $\lambda = 0$. But then $y = -1$. This isn't a boundary point. If $y = 0$ then $x = 0$ no matter what value λ takes. The point $(0, 0)$ is not a boundary point either. So, we have x and y both nonzero. This means that we can write the above equation as

$$\frac{y + 1}{x} = \frac{x}{y}.$$

Therefore $x^2 = y^2 + y$. Plugging this into the equation $g(x, y) = 0$ we get

$$2y^2 + y - 3 = 0.$$

The two solutions are $y = 1$ and $y = -3/2$. When $y = 1$, we have $x = \pm\sqrt{2}$. When $y = -3/2$, we have $x = \pm\sqrt{3}/2$. So, the candidate points are

$$(1, \sqrt{2}); \qquad (1, -\sqrt{2}); \qquad (-3/2, \sqrt{3}/2); \qquad (-3/2, -\sqrt{3}/2).$$

Evaluating f at these points, we get:

$$1 + \sqrt{2}; \qquad 1 - \sqrt{2}; \qquad -3/2 - 3\sqrt{3}/4; \qquad -3/2 + 3\sqrt{3}/4.$$

The max is $\sqrt{2} + 1$ and the min is $-3/2 - 3\sqrt{3}/4$.

4: Write $r_1(t) = (a(t), 0)$ and $r_2(t) = (b(t), c(t))$. The square distance is given by

$$f(a, b, c) = (b - a)^2 + c^2.$$

Then

$$E(t) = f(a(t), b(t), c(t)).$$

The chain rule gives

$$\frac{dE}{dt}(0) = \nabla f(1, 2, 3) \cdot (a'(0), b'(0), c'(0)).$$

Here $a' = da/dt$, etc. We have

$$\nabla f(1, 2, 3) = \Big[(2(a - b), 2(b - a), 2c)\Big]_{(1,2,3)} = (-2, 2, 6),$$

We have $a'(0) = 2$ and $b'(0) = c'(0) = 1$. The final answer is given by the dot product $(-2, 2, 6) \cdot (2, 1, 1) = 4$.