Math 20 Midterm 2 solutions

1.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 2y\ dy = \int_{-1}^{1} 1 - x^2\ dx = 4/3.$$

2. Set up the shape so that the bottom left vertex is $(0,0)$. The total mass is $3/2$. This gives us

$$\overline{x} = \frac{2}{3}\int_{y=0}^{1}\int_{x=0}^{2-y} x\ dx\ dy = 7/9.$$

$$\overline{y} = \frac{2}{3}\int_{y=0}^{1}\int_{x=0}^{2-y} y\ dx\ dy = 4/9.$$

The distance to the origin is

$$\sqrt{(4/9)^2 + (7/9)^2} = \sqrt{65}/9.$$

3a. To find there the top and bottom surfaces intersect, compute

$$x^2 + y^2 = 18 - x^2 - y^2.$$

This gives

$$\rho^2 = x^2 + y^2 = 9.$$

That is, $\rho = 3$. The region R projects into the disk $\rho \leq 3$ in the xy plane. In cylindrical coordinates, we are therefore integrating

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{r^2}^{18-r^2} (r^2\cos^2(\theta))r\ dr\ d\theta.$$

3b. The top and bottom halves of ∂R have the same area, by symmetry. So, we'll compute the area of the bottom half and then double it. The parametric equations

$$S(x,y) = (x, y, x^2 + y^2); \qquad 0 \leq x^2 + y^2 \leq 3.$$

describe the bottom half. The desired integral is

$$\int\int_{R} \sqrt{1 + 4x^2 + 4y^2} dA,$$

where R is the disk of radius 3. Switching to polar coordinates, we get the integral

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\ r\ dr\ d\theta.$$

For the inside integral, make the substitution

$$u = 1 + 4r^2; \qquad du = 8r\ dr.$$

This gives

$$A = \frac{1}{8} \int_0^{2\pi} \int_1^{37} u^{1/2} du\ d\theta = \frac{1}{8} \times 2\pi \times \frac{2}{3} \times \left(37^{3/2} - 1\right) = \frac{\pi}{6}\left(37^{3/2} - 1\right).$$

The final answer is

$$2A = \frac{\pi}{3}\left(37^{3/2} - 1\right).$$

4. First of all, we compute easily that $J_F = 6$, at all points. Also, $y = u + v$ The change of variables formula therefore gives

$$\int_R y^2\ dx\ dy\ dz = 6 \int_0^1 \int_0^1 \int_0^1 (u+v)^2\ du\ dv\ dw = 7$$

I've omitted the last calculation, which is really easy.