Math 201 Final: This final is due Tuesday, Dec 12, at noon. Please send me email if you have any questions about it. You can use the book but no other references. Do 4 of the problems.

1. Consider the paraboloid P in $\boldsymbol{R}^3$ given by the equations $z = x^2 + y^2$. Let $\gamma \subset P$ be the curve given by $\gamma(t) = (t, 0, t^2)$. Describe the behavior of the Jacobi fields along γ in as much detail as you can.

2. Construct a complete smooth Riemannian metric on $\boldsymbol{R}^2$ with the property that ∞ is the supremum of the sectional curvatures and $-\infty$ is the infimum of the sectional curvatures. In other words, the sectional curvature becomes arbitrarily negative at some points and arbitrarily positive at other points.

3. Consider the metric on $\boldsymbol{R}^3$ given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle_{x,y,t} = e^{2t} u_1 v_1 + e^{-2t} u_2 v_2 + u_3 v_3$$

Is this a metric of non-positive curvature on $\boldsymbol{R}^3$? (This is known as the *solvable metric*, because of its close connection to a certain solvable Lie group.)

4. An *ideal polyhedron* in hyperbolic n-space $\boldsymbol{H}^n$ is a polyhedron whose vertices all lie on the ideal boundary of $\boldsymbol{H}^n$. Prove that any ideal polyhedron in $\boldsymbol{H}^n$ has finite volume.

5. Let $SO(3)$ denote the Lie group of determinant 1 orthogonal 3×3 matrices, equipped with a bi-invariant Riemannian metric. Prove that $SO(3)$ has constant curvature.

6. Let X be the manifold obtained by deleting $n \geq 3$ points from the two dimensional sphere. Prove that X admits a metric of zero curvature with the following two properties: The diameter of X is finite and, for any $\epsilon > 0$, each deleted point has a neighborhood of diameter less than ϵ.

7. Given a positive smooth function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ we can put a Riemannian metric on $\boldsymbol{R}^2$ using the formula

$$\langle v_1, v_2 \rangle_p = f(p)\ v_1 \cdot v_2.$$

Compute the sectional curvature of this metric in terms of f.