# Math 181 HW

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**1.** Say that a CAT(-1) space is a complete Riemannian manifold whose sectional curvatures are all less or equal to $-1$. (This is common terminology, except that CAT(-1) usually refers to more general kinds of spaces.) Let $\epsilon > 0$ be given. Prove that there is some $R$ with the following property: Suppose $\gamma_1$ and $\gamma_2$ are two geodesic segments in $\boldsymbol{H}^2$ such that

- Both $\gamma_1$ and $\gamma_2$ have length at least $R$.
- Each endpoint of $\gamma_1$ is within 1 of some endpoint of $\gamma_2$. (That is, the endpoints are close in pairs.)

Then some point of $\gamma_1$ is within $\epsilon$ of some point of $\gamma_2$. Hints: (You probably would want to use the Rauch Comparison Theorem. As a warm-up, do this for the hyperbolic plane.)

**3.** (This is Do Carmo §7, ex 12) Prove that a homogeneous manifold is complete. A homogeneous manifold $M$ is one with the following property: For any two points in $M$ there is an isometry of $M$ taking the one point to the other.

**4.** This problem asks you to work out some of the details of the *Fubini-Study metric* on the complex projective plane. Think of $S^5$ as the unit sphere in $\boldsymbol{C}^3$, the 3 dimensional vector space of complex numbers. There is a canonical projection from $S^5$ to $\boldsymbol{CP}^2$. One simply lets $\pi(p) = [p]$, where $[p]$ is the equivalence class of points which agree with $p$ up to multiplication by a unit complex number. Each unit complex number $u$ induces an isometric action on $S^5$ by the rule $u \cdot x = ux$. Thinking of $S^1$ as the group of unit complex numbers, the above action gives a group action $S^1 : S^5 \to S^5$ whose quotient is $\boldsymbol{CP}^2$. let $\pi : S^5 \to \boldsymbol{CP}^2$ be the quotient map.

At each point $p \in S^5$ the tangent plane $T_p(S^5)$ can be decomposed as

$$T_p(S^5) = V_p \oplus V_p^{\perp},$$

where $V_p$ is tangent to the orbits of the $S^1$-action and $V_p^{\perp}$ is the orthogonal complement. Given any $[p] \in \boldsymbol{CP}^2$ and any two vectors $v_1, v_2 \in T_{[p]}(\boldsymbol{CP}^2)$ we can choose a lift $p \in S^5$ and find corresponding vectors $\widetilde{v}_1, \widetilde{v}_2 \in V_p^{\perp}$ which project to $v_1$ and $v_2$. That is, $d\pi(\widetilde{v}_j) = v_j$. We define $\langle v_1, v_2 \rangle$ to be the dot product $\widetilde{v}_1 \cdot \widetilde{v}_2$. Prove that this construction leads to a well-defined Riemannian metric on $\boldsymbol{CP}^2$ which has positive sectional curvature.

**5.** The *regular ideal octahedron* in hyperbolic 3 space is the polyhedron whose vertices are "at infinity" and arranged in the pattern of a regular Euclidean octahedron, when the ball model of hyperbolic space is used. Prove that it is possible to isometrically identify the sides of the regular ideal octahedron in order to produce a hyperbolic 3-manifold.