Math 181 Handout 16

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March 9, 2010

The purpose of this handout is to describe continued fractions and their connection to hyperbolic geometry.

1 The Gauss Map

Given any $x \in (0,1)$ we define

$$\gamma(x) = (1/x) - \text{floor}(1/x). \qquad (1)$$

Here, floor(y) is the greatest integer less or equal to y. The Gauss map has a nice geometric interpretation, as shown in Figure 1. We start with a $1 \times x$ rectangle, and remove as many $x \times x$ squares as we can. Then we take the left over (shaded) rectangle and turn it 90 degrees. The resulting rectangle is proportional to a $1 \times \gamma(x)$ rectangle.

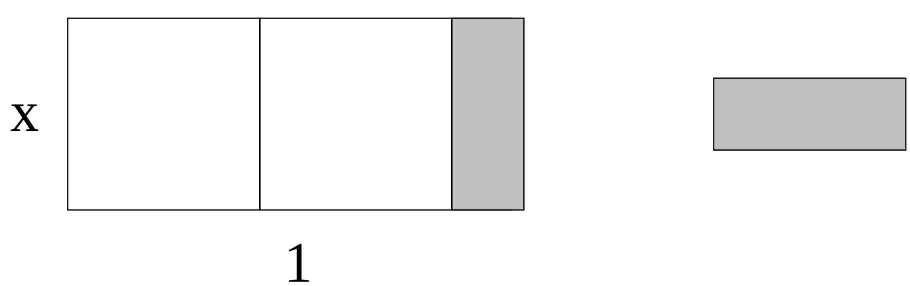

Figure 1

Starting with $x_0 = x$, we can form the sequence $x_0, x_1, x_2, ...$ where $x_{k+1} = \gamma(x_k)$. This sequence is defined until we reach an index k for which $x_k = 0$. Once $x_k = 0$, the point x_{k+1} is not defined.

Exercise 1: Prove that the sequence $\{x_k\}$ terminates at a finite index if and only if x_0 is rational. (Hint: Use the geometric interpretation.)

Consider the rational case. We have a sequence $x_0, ..., x_n$, where $x_n = 0$. We define

$$a_{k+1} = \text{floor}(1/x_k); \qquad k = 0, ..., n-1. \tag{2}$$

The numbers a_k also have a geometric interpretation. Referring to Figure 1, where $x = x_k$, the number a_{k+1} tells us the number of squares we can remove before we are left with the shaded rectangle. In the picture shown, $a_{k+1} = 2$. Figure 2 shows a more extended example. Starting with $x_0 = 7/24$, we have

- $a_1 = \text{floor}(24/7) = 3$.
- $x_1 = 24/7 - 3 = 3/7$.
- $a_2 = \text{floor}(7/3) = 2$.
- $x_2 = (7/3) - 2 = 1/3$.
- $a_3 = \text{floor}(3) = 3$.
- $x_3 = 0$.

In figure 2 we can read off the sequence $(a_1, a_2, a_3) = (3, 2, 3)$ by looking at the number of squares of each size in the picture. The overall rectangle is $1 \times x_0$.

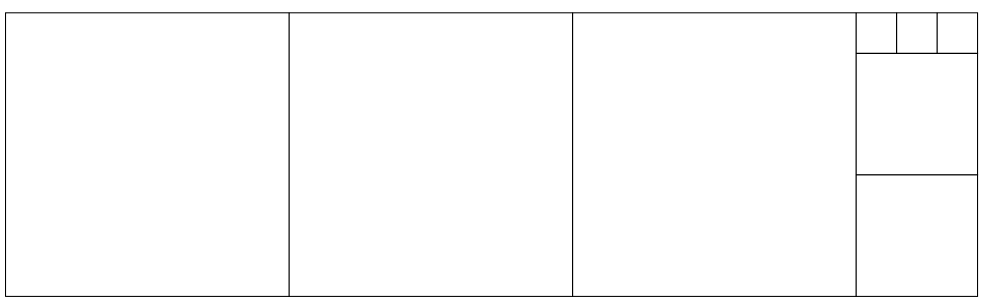

Figure 2

2 Continued Fractions

Again, sticking to the rational case, we can get an expression for x_0 in terms of $a_1, ..., a_n$. In general, we have

$$x_{k+1} = \frac{1}{x_k} - a_{k+1},$$

which leads to

$$x_k = \frac{1}{a_{k+1} + x_{k+1}}. \tag{3}$$

But then we can say that

$$x_0 = \frac{1}{a_1 + x_1} = \frac{1}{a_1 + \frac{1}{a_2 + x_2}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + x_3}}} \cdots \tag{4}$$

We introduce the notation

$$\alpha_1 = \frac{1}{a_1}; \qquad \alpha_2 = \frac{1}{a_1 + \frac{1}{a_2}}; \qquad \alpha_3 = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} \cdots \tag{5}$$

In making these definitions, we are chopping off the x_k in each expression in Equation 4. The value of α_k depends on k, but $x_0 = \alpha_n$ because $x_n = 0$.

Cosidering the example from the previous section, we have

$$\alpha_1 = \frac{1}{3}; \qquad \alpha_2 = \frac{1}{3 + \frac{1}{2}} = \frac{2}{7}; \qquad \alpha_3 = x_0 = \frac{7}{24}.$$

We say that two rational numbers p_1/q_1 and p_2/q_2 are *farey related* if

$$\det \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = p_1 q_2 - p_2 q_1 = \pm 1. \tag{6}$$

In this case, we write $p_1/q_2 \sim p_2/q_2$. For instance $1/3 \sim 2/7$ and $2/7 \sim 7/24$. This is no accident.

Exercise 2: Starting with any rational $x_0 \in (0, 1)$ we get a sequence $\{\alpha_k\}$

as above. Prove that $\alpha_k \sim \alpha_{k+1}$ for all k. Hint: induction.

Exerxise 3: Consider the sequence of differences $\beta_k = \alpha_{k+1} - \alpha_k$. Prove that the signs of β_k alternate. Thus, the sequence $\alpha_1, \alpha_2, \alpha_3, ...$ alternately over-approximates and under-approximates $x_0 = \alpha_n$.

Exercise 4: Prove that the denominator of α_{k+1} is greater than the denominator of α_k for all k. In particular, the α-sequence does not repeat. With a little bit of extra effort, you can show that the sequence of denominators grows at least exponentially.

3 The Farey Graph

Now we will switch gears and discuss an object in hyperbolic geometry. Let $\boldsymbol{H}^2$ denote theupper half-plane model of the hyperbolic plane. We form a geodesic graph $\mathcal{G}$ in $\boldsymbol{H}^2$ as follows. The vertices of the graph are the rational points in $\boldsymbol{R} \cup \infty$, the ideal boundary of $\boldsymbol{H}^2$. The point ∞ counts as rational, and is considered to be the fraction $1/0$. The edges of the graph are geodesics joining farey related rationals. For instance, the vertices

$$0 = \frac{0}{1}; \qquad 1 = \frac{1}{1}; \qquad \infty = \frac{0}{1}$$

are the vertices of an ideal triangle T_0 whose boundary lies in $\mathcal{G}$.

Let $\Gamma = PSL_2(\boldsymbol{Z})$ denote the group of integer 2×2 matrices acting on $\boldsymbol{H}^2$ by linear fractional transformations. As usual, Γ also acts on $\boldsymbol{R} \cup \infty$.

Exercise 4: Let $g \in \Gamma$ be some element. Suppose $r_1 \sim r_2$. Prove that $g(r_1) \sim g(r_2)$. In particular, g is a symmetry of $\mathcal{G}$.

Now we know that Γ acts as a group of symmetries of $\mathcal{G}$. We can say more. Suppose e is an edge of $\mathcal{G}$, connecting p_1/q_1 to p_2/q_2. The matrix

$$\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix}^{-1}$$

carries e to the edge connecting $0 = 0/1$ to $\infty = 1/0$. We call this latter edge *our favorite*. In other words, we can find a symmetry of $\mathcal{G}$ that carries any edge we like to our favorite edge. Since Γ is a group, we can find an element of Γ carrying any one edge e_1 of $\mathcal{G}$ to any other edge e_2. We just compose the element that carries e_1 to our favorite edge with the inverse of the element that carries e_2 to our favorite edge. In short Γ acts transitively on the edges of $\mathcal{G}$.

Exercise 5: Prove that no two edges of $\mathcal{G}$ cross each other. (Hint: By the symmetry we have just discussed, it suffices to prove that no edge crosses our favorite edge.

We have exhibited an ideal triangle T_0 whose boundary lies in $\mathcal{G}$. Our favorite edge is an edge of this triangle. It is also an edge of the ideal triangle T_1 with vertices

$$\frac{0}{1}; \qquad \frac{1}{0}; \qquad \frac{-1}{1}.$$

The boundary of this triangle lies in $\mathcal{G}$ as well. Thus, our favorite edge is flanked by two ideal triangles whose boundaries lie in $\mathcal{G}$. But then, by symmetry, this holds for every edge of $\mathcal{G}$. Starting out from T_0 and moving outward in a tree-like manner, we recognize that $\mathcal{G}$ is the set of edges of a triangulation of $\boldsymbol{H}^2$ by ideal triangles. Figure 2 shows a finite portion of $\mathcal{G}$. The vertical line on the left is our favorite line. The vertical line on the right connects 1 to ∞.

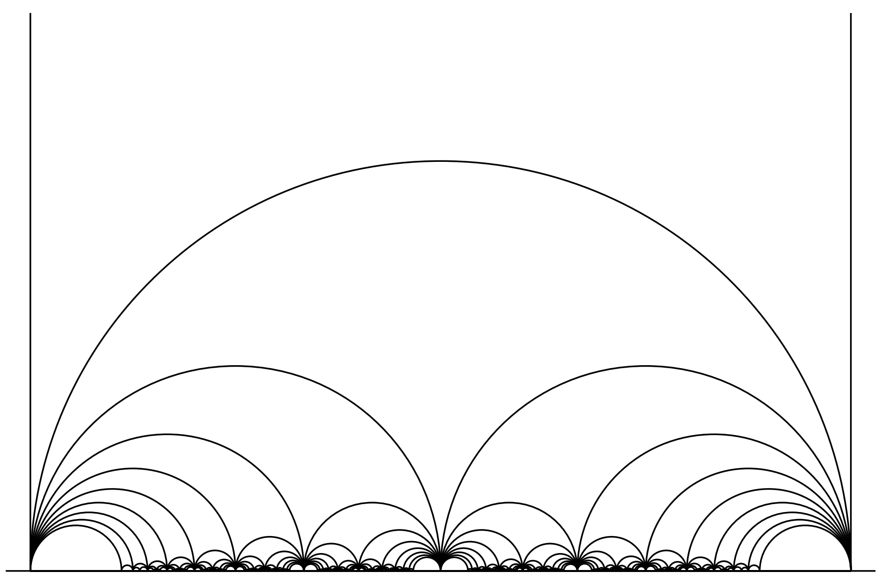

Figure 3

4 Continued Fractions and the Farey Graph

Let's go back to continued fractions and see how they fit in with the Farey graph. Let $x_0 \in (0,1)$ be a rational number. We have the sequence of approximations $\alpha_1, ..., \alpha_n = x_0$ as in Equation 5. It is convenient to also define

$$\alpha_{-1} = \infty; \qquad \alpha_{-0} = 0; \tag{7}$$

If we consider the larger sequence $\alpha_{-1}, ..., \alpha_n$, the statements of Exercises 2 and 3 remain true. In particular, we have a path $P(x_0)$ in the Farey graph that connects ∞ to x_0, obtained by connecting ∞ to 0 to α_1, etc. The example gave above doesn't produce such a nice picture, so we will give some other examples.

Let $x_0 = 5/8$. This gives us

$$a_1 = ... = a_5 = 1$$

and

$$\alpha_1 = 1; \qquad \alpha_2 = \frac{1}{2}; \qquad \alpha_3 = \frac{2}{3}; \qquad \alpha_4 = \frac{3}{5}; \qquad \alpha_5 = x_0 = \frac{5}{8}.$$

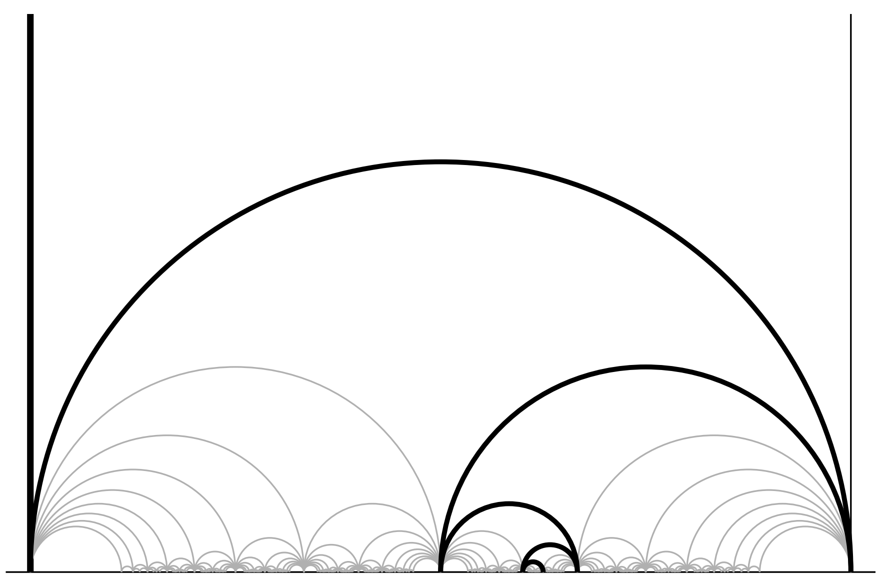

Figure 3

Taking $x_0 = 5/7$ gives

$$a_1 = 1; \qquad a_2 = 2; \qquad a_3 = 2.$$

and

$$\alpha_1 = 1; \qquad \alpha_2 = \frac{2}{3}; \qquad \alpha_3 = \frac{5}{7}.$$

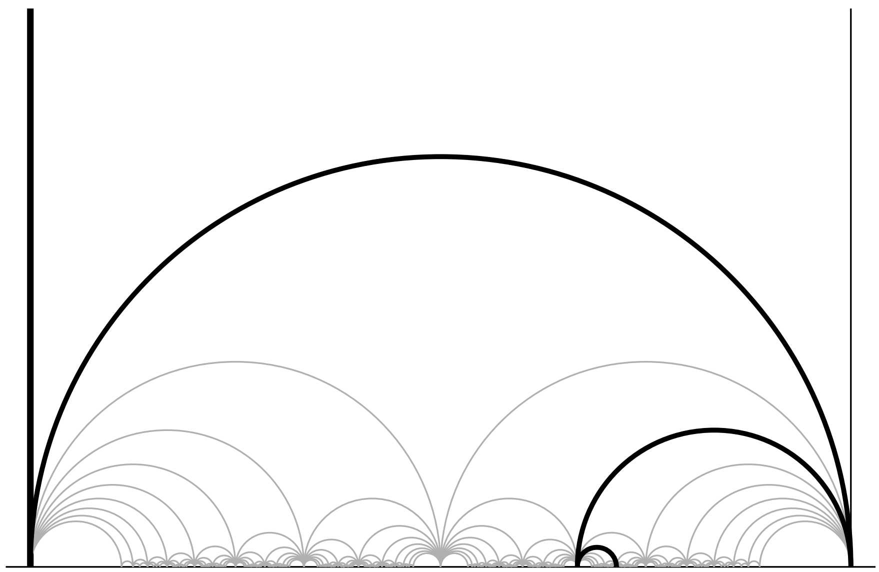

Figure 4

There are three things we would like to point out about these pictures. First, they make a zig-zag pattern. This always happens, thanks to Exercises 3 and 4 above. Exercise 4 says that the path cannot backtrack on itself, and then Exercise 4 forces the back-and-forth behavior.

Second, we can read off the numbers $a_1, ..., a_n$ by looking at "the amount of turning" the path makes at each vertex. In Figure 4, our path turns "one click" at α_0, then "two clicks" at α_1, then "two clicks" at α_2. This corresponds to the sequence $(1, 2, 2)$. Similarly, the path in Figure 3 turns "one click" at each vertex, and this correspons to the sequenc $(1, 1, 1, 1, 1)$.

Exercise 6: Prove that the observation about the turns holds for any rational $x_0 \in (0, 1)$.

Third, the diameter of the kth arc in our path is less than $1/k(k-1)$. This is a terrible estimate, but it will serve our purposes below. To understand this estimate, note that the kth arc connects $\alpha_{k-1} = p_{k-1}/q_{k-1}$ to $\alpha_k = p_k/q_k$, and $\alpha_{k-1} \sim \alpha_k$. The diameter of the kth arc is

$$|\alpha_{k-1} - \alpha_k| = \left|\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}\right| =^* \frac{1}{q_{k-1}q_k} \leq \frac{1}{k(k-1)}.$$

The starred equation comes from the fact that α_{k-1} and α_k are Farey-related. The last inequality comes from Exercise 4. As we mentioned in Exercise 4, the denominators of the α-sequence grow at least exponentially. So, actually, the arcs in our path shrink exponentially fast.

5 The Irrational Case

So far, we have concentrated on the case when x_0 is rational. If x_0 is irrational, then we produce an infinite sequence $\{\alpha_k\}$ of rational numbers that approximate x. From what we have said above, we have

$$x \in [\alpha_k, \alpha_{k+1}] \quad \text{or} \quad x \in [\alpha_{k+1}, \alpha_k] \tag{8}$$

for each index k, with the choice depending on the parity of k, and also

$$\lim_{k\to\infty} |\alpha_k - \alpha_{k+1}| = 0. \tag{9}$$

Therefore

$$x_0 = \lim_{k\to\infty} \alpha_k. \tag{10}$$

The corresponding infinite path in the Farey graph starts at ∞ and zig-zags downward forever, limiting on x.

The nicest possible example is probably

$$x_0 = \frac{\sqrt{5}-1}{2} = 1/\phi,$$

where ϕ is the golden ratio. In this case, $a_k = 1$ for all k and α_k is always ratio of two consecutive Fibonacci numbers. The path in this case starts out as in Figure 3 and continues the pattern forever. Taking some liberties with the notation, we can write

$$\frac{1}{\phi} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

Since $\phi = 1 + (1/\phi)$ we can equally well write

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} \tag{11}$$

This is a truly famous equation!

The $\{a_k\}$ sequence is known as the *continued fraction expansion* of x_0. In case $x_0 > 1$, we pad the sequence with floor(x_0). So, $1/\phi$ has continued fraction expansion $1, 1, 1, ...$ and ϕ has continued fraction expansion $1; 1, 1, 1...$ The subject of continued fractions is a vast one. Here I'll mention a few facts.

- An irrational number $x_0 \in (0, 1)$ is the root of an integer quadratic equation $ax^2 + bx + c = 0$ if and only if it has a continued fraction expansion that is eventually periodic.

- The famous number e has continued fraction expansion

$$2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10 \dots$$

- The continued fraction expansion of π is not known.

In spite of having a huge literature, the subject of continued fractions abounds with unsolved problems. For instance, it is unknown of the $\{a_k\}$ sequence for the cube root of 2 is unbounded. In fact, this is unknown for any root of an integer polynomial equation that is neither quadratic irrational nor rational.

We close this chapter with a sketch-proof of the statement about about quadratic irrationals. To simplify the proof, we will prove the somewhat different statement that $x_0 \in (0, 1)$ has periodic continued fraction expansion if and only if x_0 is the fixed point of an element of $\Gamma = PSL_2(\boldsymbol{R})$.

Suppose that x_0 has an eventually-periodic continued fraction expansion. Let $P(x_0)$ be the zig-zag path joining ∞ to x_0.

Exercise 7: Prove that the path $P(x_0)$ eventually is periodic, in the sense that there is a finite union P' of edges of $P(x_0)$ and an element $g \in \Gamma$ such that the union $\bigcup g^k(P')$ is a partition of the "tail end" of $P(x_0)$

Note that x_0 is a fixed point of the element g from Exercise 7. This proves one half of the claim.

Exercise 8: Let x and y be two distinct irrational points in $\boldsymbol{R} \cup \infty$. Let β be the geodesic connecting x to y. Prove that there is a unique path in the Farey graph that connects x to y and only uses edges that cross β. Call this path $P(x, y)$.

Exercise 9: Prove that $P(x_0, y)$ and $P(x_0)$ agree in a neighborhood of x_0. Here $P(x)$ is the zig-zag path connecting ∞ to x_0, constructed from the continued fraction expansion of x_0.

The element g, being a loxodromic isometry, has a second fixed point y_0. The canonical path $P(x_0, y_0)$ is invariant under g because it is canonical. Hence $P(x_0, y_0)$ is periodic in a geometric sense. But $P(x_0, y_0)$ agrees with $P(x_0)$ in a neighborhood of x_0. But then the tail end of $P(x_0)$ is periodic. But one can read off the continued fraction expansion from the geometry of $P(x_0)$. Hence, the continued fraction expansion of x_0 is eventually periodic.