**Math 52 Sample Midterm Solutions**

**1.** This is just a straight computation using Gauss-Jordan elimination. The final answer is:

$$x_1 = \frac{-30 - 67x_4}{33}; \qquad x_2 = \frac{9 + 8x_4}{11}; \qquad x_3 = \frac{324 + 475x_4}{99}.$$

Here $x_4$ is the free variable. (The actual midterm probably won't have anything this messy.)

**2.** Let $M = M_0$. Expanding by cofactors across the first row, we see that $\det(M_0) = \det(M_1)$, where

$$M_1 = \begin{matrix} 1 & a & 0 & 0 \\ b & 1 & c & 0 \\ 0 & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Expanding by cofactors across the bottom row, see that $\det(M_1)$ equals $\det(M_2)$ where

$$M_2 = \begin{matrix} 1 & a & 0 \\ b & 1 & c \\ 0 & d & 1 \end{matrix}$$

It is pretty easy to work out that

$$\det(M_2) = 1 - ac - bd.$$

So, you just have to choose 4 distinct and nonzero values of $a, b, c, d$ so that $ac + bd = 1$. You could take $a = 10$ and $b = 100$ and $c = 1/20$ and $d = 1/200$ for instance.

**3.** Let's find the eigenvalues first. To so this, we solve

$$\det(M - tI) = 0.$$

This leads to

$$2 + 3t - t^3 = 0.$$

This polynomial factors as

$$-(t + 1)^2(t - 2)$$

so the roots are $t = -1$ and $t = 2$. These are the eigenvalues.

Let's first consider the case when $t = 2$. Here is the matrix $M - 2I$:

$$M - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

One can find the solutions to $(M-2I)x = 0$ using Gauss-Jordan elimination, and it turns out that the vector $(r, r, r)$ is the only kind of solution you can have. You can pretty much see this directly, because the sum of entries in any row of $M$ equals zero.

Here is the matrix $M - (-1)I = M + I$:

$$M + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

If $x$ is any vector then $(M + I)x = x_1 + x_2 + x_3$, so $x$ is an eigenvector corresponding to the eigenvalue 1 provided that $x_1 + x_2 + x_3 = 0$.

So, in summary, here is the answer:

- One of the eigenvalues is 2. The corresponding eigenvectors have the form $(r, r, r)$, where $r$ is any nonzero number.

- The other eigenvalue is $-1$. The corresponding eigenvectors have the form $(a, b, c)$, where $a + b + c = 0$.

**4:** (*Note: In this problem I meant to say explicitly that $M$ is invertible. I hope that this was clear from the context.*)

It turns out that $(AB)^t = B^t A^t$ for any two square matrices. Assuming this general fact, note that

$$(M^{-1})^t M^t = (MM^{-1})^t = I^t = I.$$

This shows that multiplying $M^t$ on the left by $(M^{-1})^t$ gives the identity. On the other hand, multiplying $M^t$ on the left by $(M^t)^{-1}$ gives the identity. Therefore

$$(M^{-1})^t \times M^t = (M^t)^{-1} \times M^t.$$

Now multiply both sides of this equation on the right by $(M^t)^{-1}$. This gives

$$(M^{-1})^t = (M^{-1})^t \times M^t \times (M^t)^{-1} = (M^t)^{-1} \times M^t \times (M^t)^{-1} = (M^t)^{-1}.$$

This establishes the result, assuming the general fact.

Probably I would give you credit for a problem like this if you just stated the general fact and mentioned that it was proved in the book. However, a complete answer might indicate why the general fact was true. Here is the explanation.

Let $M_{ij}$ denote the $(i, j)$th entry of any matrix. By definition of matrix multiplication, we have

$$(MN)_{ij} = \sum_k M_{ik} N_{kj}.$$

Using this fact, and the definition of transpose, we have

$$(AB)^t_{ij} =$$

$$(AB)_{ji} =$$

$$\sum_k A_{jk} B_{ki} =$$

$$\sum_k (A^t)_{kj} (B^t)_{ik} =$$

$$\sum_k (B^t)_{ik} (A^t)_{kj} = (B^t A^t)_{ij}.$$

This shows that $(AB)^t$ and $B^t A^t$ have the same general $(i, j)$ th entry. Hence, they are equal.