The Quaternions

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The purpose of this handout is to introduce the quaternions and discuss some of their basic properties.

1 Basic Definitions

To define the quaternions, we first introduce the symbols i, j, k. These symbols satisfy the following properties:

$$i^2 = j^2 = k^2 = -1; \qquad ij = k; \qquad jk = i; \qquad ki = j. \tag{1}$$

Also, for any real number x, we have

$$ix = xi; \qquad jx = xj; \qquad kx = xk. \tag{2}$$

You can work out other rules from these properties. For example, suppose you want to compute the mystery symbol $T = ji$. Note that

$$Ti = jii = j(-1) = (-1)j = -j = -ki.$$

Cancelling the i gives $T = -k$. In short, $ji = -k$. The other rules are

$$ji = -k; \qquad kj = -i; \qquad ik = -j. \tag{3}$$

A *quaternion* is an object of the form $a + bi + cj + dk$, where a, b, c, d are real numbers.

Quaternions are added componentwise and multiplied using the "foil method". For addition

$$(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) =$$

$$(a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k. \tag{4}$$

To do the multiplication, you expand out the product

$$(a_1 + b_1 i + c_1 j + d_1 k) \times (a_2 + b_2 i + c_2 j + d_2 k)$$

as you would for a polynomial and then simplify all the terms involving ij, ik, etc., using the rules above. For instance

$$(3i + j) \times (7j + 2k) =$$

$$21ij + 6ik + 7jj + 2jk =$$

$$21k - 6j - 7 + 2i = -7 + 2i - 6j + 21k.$$

Problem 1: Show that $(q_1 q_2)q_3 = q_1(q_2 q_3)$ for any three quaternions q_1, q_2, q_3. That is, the multiplication is associative.

2 Conjugates and Norms

Given a quaternion $q = a + bi + cj + dk$, we have the conjugate

$$\overline{q} = a - bi - cj - dk. \tag{5}$$

Problem 2: Show that

$$q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2. \tag{6}$$

The *norm* of q is defined to be

$$|q| = \sqrt{q\overline{q}}. \tag{7}$$

q is called a *unit quaternion* if $|q| = 1$. In case q is a unit quaternion, note that $\overline{q}$ has the property that $q\overline{q} = \overline{q}q = 1$. In other words, $\overline{q} = 1/q$. In general, we have the division formula

$$\frac{q_1}{q_2} = \frac{q_1 \overline{q}_2}{|q_2|^2}. \tag{8}$$

This works as long as $|q_2| \neq 0$.

problem 3: Show that

$$\overline{pq} = \overline{q} \times \overline{p}, \tag{9}$$

for any two quaternions p and q.

Given the calculation from problem 3, we have

$$|pq|^2 = pq \times \overline{pq} = p \times q \times \overline{q} \times \overline{p} = p \times |q|^2 \times \overline{p} = p\overline{p} \times |q|^2 = |p|^2|q|^2.$$

Taking square roots of both sides, we get

$$|pq| = |p||q|. \tag{10}$$

This holds for any two quaternions.

3 The Three Dimensional Sphere

Let S^3 denote the set of all unit quaternions. The equation for S^3 is given by

$$a^2 + b^2 + c^2 + d^2 = |q|^2 = 1. \tag{11}$$

This is the equation for the unit sphere in four dimensional space.

Now let's verify that S^3 is a group, with the multiplication law. We need to check the 4 basic axioms.

1. If $p, q \in S^3$, then so is pq. This is a special case of Equation 10.
2. $(pq)r = (p(qr)$. This is Problem 1 above.
3. 1 is a unit quaternion and satisfies $1q = q1$ for all $q \in S^3$.
4. Let $q^{-1} = \overline{q}$. Then $qq^{-1} = q\overline{q} = |q|^2 = 1$. Likewise $q^{-1}q = 1$.

This verifies all the group laws.

4 Representing Rotations by Quaternions

This section is somewhat more advanced than previous sections.

A quaternion of the form $0+bi+cj+dk$ is called *pure*. Let V denote the set of pure quaternions. If you know about linear algebra, you will recognize that V is a 3 dimensional real vector space, that we are identifying with $\boldsymbol{R}^3$. If you don't know what this means, you can just think informally that V is a copy of $\boldsymbol{R}^3$.

Exercise 4: Suppose that q is a unit quaternion and p is a pure quaternion. Prove that qpq^{-1} is another pure quaternion.

Given a unit quaternion q, define

$$T_q(p) = qpq^{-1}. \tag{12}$$

Exercise 4 shows that T_q is a map from V to V. Note that

$$|T_q(p)| = |q| \times |p| \times |q^{-1}| = 1 \times |p| \times 1 = |p|. \tag{13}$$

This comes from Equation 10.

Exercise 5: Let $r \in \boldsymbol{R}$ and let p_1, p_2 be real quaternions. Prove that

$$T_q(rp_1 + p_2) = rT_q(p_1) + T_q(p_2). \tag{14}$$

Exercise 5 shows that T_q is a linear map. Equation 13 shows that T_q is an isometry. This means that $\det(T_q) = \pm 1$. When $q = 1$, the map T_q is the identity map, and has determinant 1. Also, the determinant is a continuous function of q. Hence

$$\det(T_q) = 1 \tag{15}$$

for all unit quaternions q. All this information together shows that T_q acts as a rotation of 3-dimensional space.

Exercise 6: Show that every rotation of $\boldsymbol{R}^3$ (which fixes $(0,0,0)$) has the form T_q for some unit quaternion q. Also, show that $T_q = T_r$ if and only if $q = \pm r$.

The group of all rotations of $\boldsymbol{R}^3$ is denoted by $SO(3)$. We have just exhibited a map $S^3 \to SO(3)$. This map is (by Exercise 6) $2-1$ and onto. It is known as the *spin cover*.