Symmetry Groups of Platonic Solids

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September 17, 2007

The purpose of this handout is to discuss the symmetry groups of Platonic solids.

1 Basic Definitions

Let $\boldsymbol{R}^3$ denote 3-dimensional space.

A *rotation* of $\boldsymbol{R}^3$ is any map from $\boldsymbol{R}^3$ to $\boldsymbol{R}^3$ that preserves distances, fixes $(0,0,0)$, and preserves orientation. Informally, a rotation of space is any way of picking up space and rotating it about the origin. Any rotation of space induces a rotation of the unit 2-sphere centered at the origin.

If you know linear algebra, we can say that a rotation is a linear transformation $\boldsymbol{R}^3$ that is an isometry and has determinant 1.

For what it is worth, any rotation of $\boldsymbol{R}^3$ can be written as a product of matrices

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & \sin(\gamma) \\ 0 & -\sin(\gamma) & \cos(\gamma) \end{bmatrix} \tag{1}$$

The angles (α, β, γ) are essentially the *Euler angles* of the rotation. Geometrically, you get a general rotation by rotating various amounts about the 3-coordinate axes in succession.

The second handout explains an alternative way to represent rotations using quaternions.

Given any set S, a *rotation of* S is a rotatioon of $\boldsymbol{R}^3$ that maps S to itself. Let $G(S)$ denote the set of rotations of S. Note that $G(S)$ forms a group. The group law is just "composition" of the maps. $G(S)$ is called *the rotation symmetry group* of S.

2 Permutation Groups

In order to understand the rotation symmetry groups of the platonic solids, we first need to understand something about permutations. A *permutation of N things* is a map $f : \{1, ..., N\} \to \{1, ..., N\}$ which is one-to-one and onto. In otherwords, a permutation of N things is just a description (using the languages of maps) of how to rearrange the elements of the set $\{1, ..., N\}$.

Let S_N denote the set of permutations of N things. Given permutations f and g in S_N we define the permutation $f * g$ by the rule $f * g(k) = f(g(k))$. This is just the usual composition rule for functions.

Problem 1: Prove that $f * g$ is again a permutation.

The identity permutation e is just defined by the rule $e(k) = k$ for all k. The inverse of a permutation is f is given by the rule $f^{-1}(f(x)) = x$. This rule completely determines f^{-1} because f is one-to-one and onto.

Problem 2: Prove that f^{-1} is again a permutation.

We let $R(f)$ denote the number of pairs $a, b \in \{1, ..., N\}$ such that $a < b$ and $f(a) > f(b)$. We say that f is an *even permutation* if $R(f)$ is even. Otherwise we call f an *odd permutation*. Let A_N denote the set of even permutations.

Lemma 2.1 *The product of two even permutations is even. Also, half the permutations are even.*

Proof: See §5.1 below. ♠

Lemma 2.2 *The inverse of an even permutation is an even permutation.*

Proof: If f reverses the pair (a, b) then f^{-1} reverses the pair (a', b'), where $a' = f(a)$ and $b' = f(b)$. Likewise, if f^{-1} reverses (a', b') then f reverses (a, b). From this we see that $R(f) = R(f^{-1})$. ♠

The two results above show that A_N forms a group in its own right. From Lemma 2.1, we see that A_N has $N!/2$ elements.

3 Rotations of the Tetrahedron

Let T denote the regular tetrahedron centered at the origin in $\boldsymbol{R}^3$. We can take the vertices of T to be

$$(1,1,1); \qquad (1,-1,-1); \qquad (-1,1,-1); \qquad (-1,-1,1). \tag{2}$$

You can check that all these vertices are all the same distance from the origin, and also all the same distance from each other.

Each element of $G(T)$ permutes the vertices of T. Labelling the vertices $1,2,3,4$, we see that one element has the form

$$f(1)=1; \qquad f(2)=3; \qquad f(3)=4; \qquad f(4)=2. \tag{3}$$

Here $R(f)=2$ The two pairs $(2,4)$ and $(3,4)$ are reversed. So, f is an even permutation. Geometrically, you effect the element f by twirling T about its first vertex. There are 9 elements having a similar form, and they are all even, for the same reasons.

The remaining 3 elements are obtained by rotating T about a line that goes through the midpoints of a pair of opposite sides. Picture putting T in a stick, like a shish-kebob, and then twirling the stick. One of these permutations, g, has the effect:

$$g(1)=2; \qquad g(2)=1; \qquad g(3)=4; \qquad g(4)=3. \tag{4}$$

Again $R(g)=2$, so g is even. The other two elements have the same description. Note that A_4 has 12 even permutations, and we have just exhibited 12 different even permutations of 4 things obtained by applying different rotational symmetries of T. Therefore $G(T)=A_4$.

4 Rotations of the Cube

Let Q be a cube centered at $(0,0,0)$. We can take the vertices of Q to be the points $(\pm 1,\pm 1,\pm 1)$, with all 8 choices of sign. The points p and $-p$ are opposite corners of Q. There are 4 pairs of opposite corners, and these *pairs* are permuted by any rotation of Q. Say that a *corner pair* is a pair of opposite corners.

Any rotation of Q gives rise to a permutation of the corner pairs, and hence an element of S_4. Therefore, we get a map

$$\phi : G(Q) \to S_4. \tag{5}$$

Note that ϕ respects the group laws: The composition of two rotations permutes the corner pairs by composing the two permutations obtained from each of the rotations separately. In short

$$\phi(gh) = \phi(g) * \phi(h). \tag{6}$$

Problem 3: Prove that ϕ is one-to-one. In other words, if g_1 and g_2 are two rotations that permute the pairs of corners of Q in the same way, then $g_1 = g_2$.

In class we saw that $G(S)$ has 24 elements. Also, S_4 has 24 elements. The map ϕ is a one-to-one map from one set of 24 elements to another set of 24 elements. Hence ϕ is onto. This is an example of the famous *Pidgeonhole Principle*. You are sticking 24 messages into 24 pidgeonholes, in such a way that no two messages go in the same hole. From this, you see that every hole gets a message.

The map ϕ is an isomorphism from $G(Q)$ to S_4. In short, $G(Q) = S_4$.

The Octahedron: You can inscribe a cube into an octahedron in such a way that each vertex of the cube is at the center of a face of an octahedron. From this, we see that the cube and the octahedron have the same symmetry groups.

5 Rotations of the Icosahedron

Let I be the icosahedron.

As we saw in class, we can group the faces of I into 5 groups of 4, such that the centers of the faces in each group are the vertices of a regular tetrahedron. In short, we can inscribe 5 regular tetrahedra into I, such that each tetrahedron has its vertices at the centers of some faces of I.

Let $T_1, ..., T_5$ denote these tetrahedra. Each rotation of I gives a permutation of the tetrahedra, and hence an element of S_5. This gives us a map

$$\phi : G(I) \to S_5. \tag{7}$$

Again, this map respects the group laws.

Let f_1 be a face of I. The face f_1 determines the tetrahedron T_1. Let f_2 be the opposite face. Then f_2 determines another tetrahedron T_2. There are 3 elements of $G(I)$ that preserve f_1 and f_2. (That is, map them to themselves.) Let g be one of these. Then $g' = \phi(g)$ has the following effect on the tetrahedra:

$$g'(1) = 1; \qquad g'(2) = 2; \qquad g'(3) = 4; \qquad g'(4) = 5; \qquad g'(5) = 3. \tag{8}$$

From this we see that $R(g') = 2$. Hence $\phi(g)$ is an even permutation in this case. Call g a *special rotation*.

Problem 4: Prove that any rotation of I is some finite product of the special rotations.

Let $\gamma \in G(I)$ be arbitrary. Then γ is the product of special rotations. But then $\phi(\gamma)$ is the product of even permutations. Hence $\phi(\gamma)$ is an even permutation.

Now we see that ϕ maps $G(I)$ into A_5. Both $G(I)$ and A_5 have 60 elements. For the same reasons as for the octahedron, we see that ϕ is an isomorphism from $G(I)$ to A_5.

The Dodecahedron: The dodecahedron and the icosahedron are related in the same way that the cube and the octahedron are related. Hence, the rotation symmetry group of the dodecahedron is again A_5.

5.1 Proof of Lemma 2.1

Say that a *special permutation* is a permutation that swaps two consecutive integers and does nothing to the others. Let E_k be the special permutation that swaps k and $k+1$. In S_N there are $N-1$ special permutations, namely $E_1, ..., E_{N-1}$.

Problem 5: Prove that any permutation in S_N is the product of special permutations.

Problem 6: Suppose that E is a special permutation and f is a general permutation. Show that $R(Ef) = R(f) \pm 1$.

Lemma 5.1 *f is an even (or odd) permutation if and only if f can be written as the product of an even (or odd) number of special permutations.*

Proof: We write

$$f = E_{n_1} ... E_{n_k}.$$

We have $R(E_{n_1}) = 1$. But then $R(E_{n_1}E_{n_2})$ is even, by Problem 6. But then $R(E_{n_1}E_{n_2}E_{n_3})$ is odd, by Problem 6. Continuing in this way, we see that $R(f)$ has the same parity as k. ♠

If f and g are two even permutations, then we can write f and g each as the product an even number of special permutations. But then we can write fg as an even number of special permutations. Hence fg is even. This completes the proof of the first statement of Lemma 2.1.

For the second statement, let E be some special permutation. For every even permutation f, the permutation Ef is odd. The map $f \to Ef$ gives a map from the set of even permutations to the set of odd permutations. It is easy to check that this map is one-to-one and onto. In short, there are the same number of even permutations as odd ones. Hence, half the permutations are even.