Penrose Tiling Basics

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November 5, 2007

The purpose of these notes is to give an account of some of the basic properties of the Penrose tiles. None of the material here is new; all of it can be found in a variety of sources.

1 Basic Definitions

There are two Penrose tiles, a *kite* and a *dart*. Both tiles can be constructed from a regular pentagon, as shown in Figure 1.

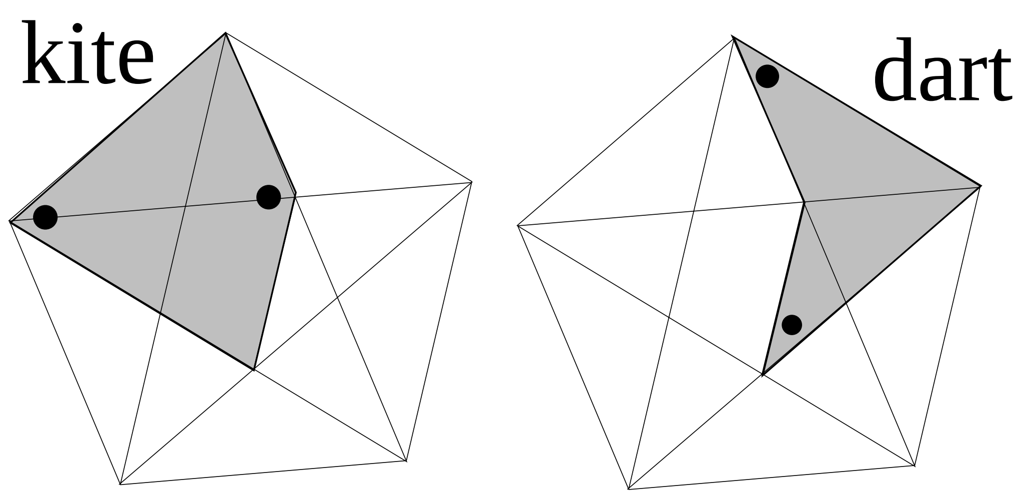

Figure 1

The edges of the kite and the dart are labelled so as to disallow certain ways of putting together the tiles. The rule is simply that the tiles can only be put together so that edges of the same length are matched and the black dots match the black dots. Figure 2 shows some examples.

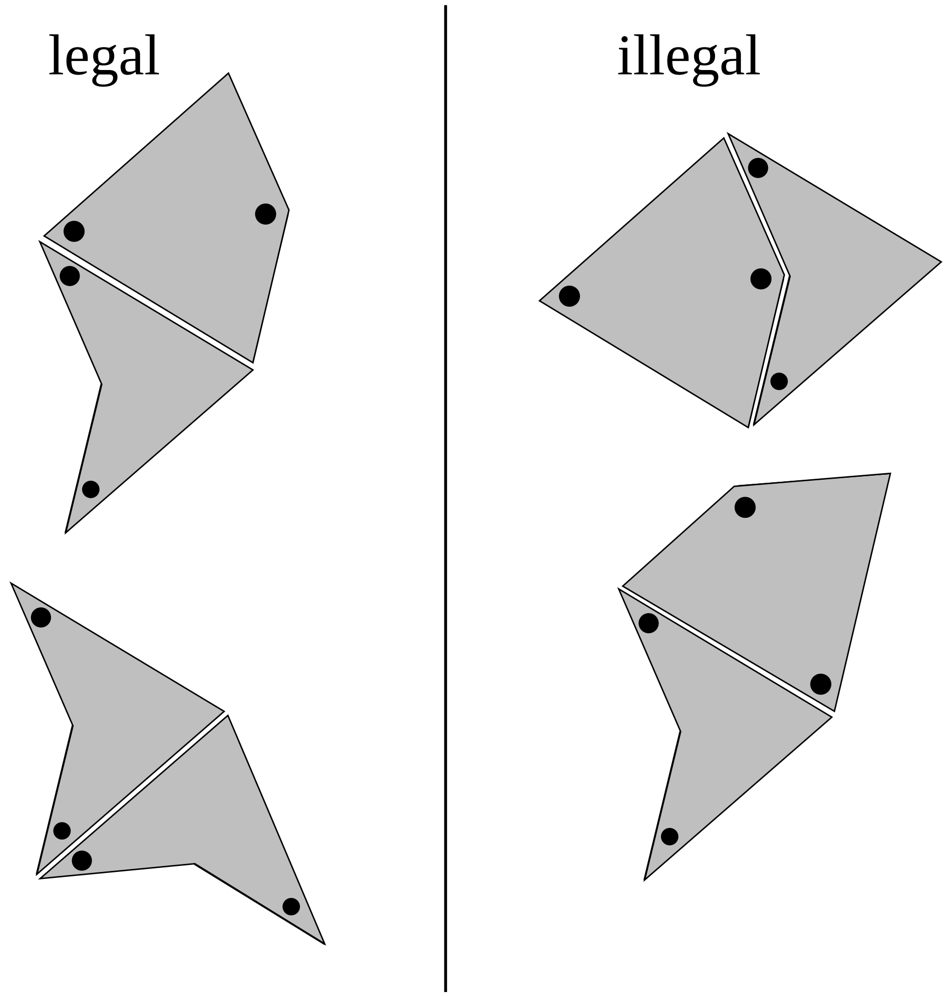

Figure 2

2 Existence of Penrose Tilings

In this secton I'll explain why one can tile the plane with Penrose tiles. Here are three observations.

- One can replace a kite by a union of two darts and two kites. The original kite is the union of two smaller kites and two "half-darts".
- One can replace a dart by a union of one dart and two kites. The original dart is the union of one kite and two "half-darts".

- If one performs the replacement process for a finite collection of Penrose tiles arranged in a legal configuration, one gets a new collection of Penrose tiles arranged in a legal configuration. The proof of this fact just amounts to a short case-by-base check for pairs of tiles.

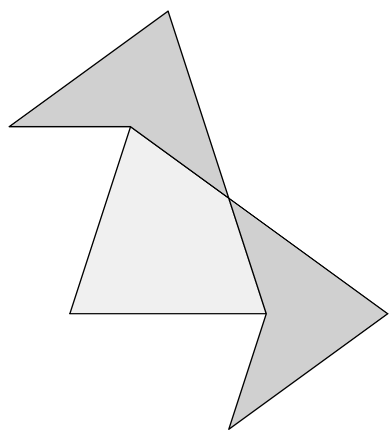

Figure 3

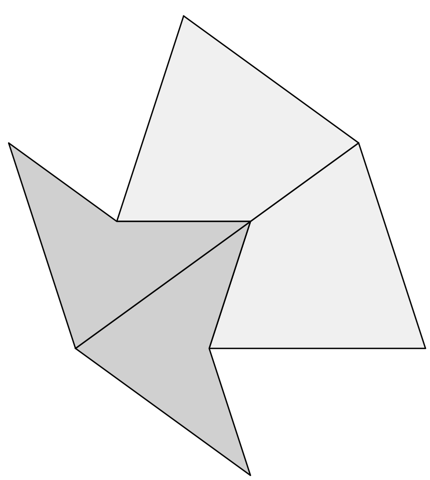

Figure 4

You can now obtain tilings of the plane by increasingly large (and fat) regions of the plane by iterating the subdivision process. Figure 5 shows what happens when each of the tiles in Figure 4 is subdivided into the replacement tiles.

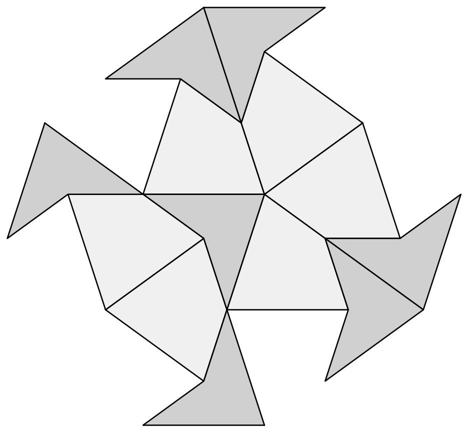

Figure 5

Figure 6 shows what happens when each of the tiles in Figure 5 is subdivided into the replacement tiles.

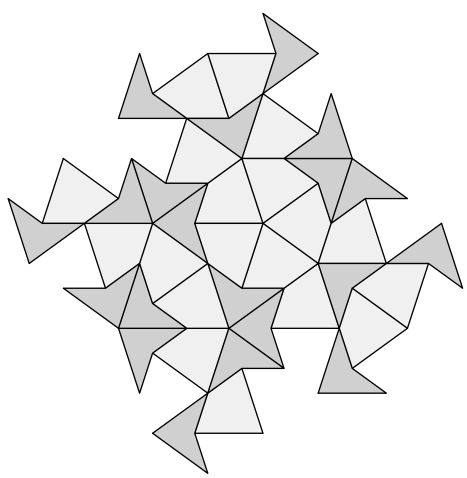

Figure 6

This process continues forever. Suppose that you always rescale the tiles so that they have the same size. Let $T_1, T_2, T_3, \ldots$ be the sequence of partial tilings you get. So, T_1 is the Penrose kite and T_{n+1} is obtained from T_n by subdividing and then dilating the picture so that the tiles are the same size as in T_n.

We would like to take a kind of limit and produce a tiling T_∞ that fills the entire plane. Here is the idea. Let B denote any disk in the plane–let's say the disk of radius 100000. There are only finitely many possible patterns for the intersection $T_n \cap B$. In other words, if we just look at the tiles inside B, then we will only see finitely many different patterns as n varies.

So, we can choose some pattern for $T_n \cap B$ that we see infinitely often. We throw out all the tilings that don't give this pattern, and we still have an infinite supply. Now we repeat the same procedure on the disk $2B$ that is concentric with B and twice as large. From amongst our infinite supply of partial tilings, we can find a (possibly) smaller infinite supply that makes the same pattern on the bigger disk $2B$. Now we throw away all the T_n that don't make this pattern on $2B$ and we continue, using the disk $4B$. Continuing this way indefinitely, we produce a legal tiling of the whole plane, layer by layer.

Now, if you have a Penrose tiling P, you can perform the subdivision process to all the tiles of P at the same time. This gives you a new Penrose tiling P'. Let's call P' the *subdivision* of P'.

3 Undoing the Subdivision

Now suppose that P' is a Penrose tiling. You can group the tiles of P' together, **in a unique way**, according to Figures 3 and 4. To perform the grouping, start with a kite K. There are two possibilities:

1. K is adjacent to two darts, D_1 and D_2, each of which shares a short edge with K. The three tiles K and D_1 and D_2 are then grouped together, as in Figure 3.

2. K is adjacent to another kite K' and one dart D along the short edges of K. Then K' is adjacent to K and another dart D' along the short edges of K'. The tiles K and K' and D and D' are then grouped together, as in Figure 4.

Now perform the above procedure for every kite of P'. Once you finish this grouping procedure, you will have grouped all the tiles of P' into groups of 3 and 4.

Now observe that each group of tiles is the subdivision of a larger Penrose tile. You can construct a new Penrose tiling P by replacing each group of tiles in P' by the corresponding larger Penrose tile that has the given group of tiles as its subdivision. In short, you can undo the subdivision process in a unique way to find the Penrose tiling P such that P' is the subdivision of P.

4 No Infinite Symmetries

We have already seen that one can tile the plane with Penrose tiles. The result in this section shows that one cannot make a "repeating pattern" out of the Penrose tiles.

Theorem 4.1 *Let P be a Penrose tiling. Suppose that the group of symmetries of P is infinite.*

Proof: The proof goes by contradiction. Suppose that the group of symmetries of P is infinite. In this case, there must be some translation T such that $T(P) = P$. We will show that this is impossible.

We know that P is the subdivision of a larger Penrose tiling P^{-1}. Likewise, P^{-1} is the subdivision of a larger Penrose tiling P^{-2}, etc. From the nature of the construction described in the previous section, we have $T(P^{-n}) = P^{-n}$ for any choice of n. However, once n is large, the tiles of P^{-n} are larger than the translation distance of T. But then it is impossible for $T(P^{-n}) = P^{-n}$. The problem is that T doesn't move points far enough to map the huge tiles of T^{-n} to themselves. ♠

5 Five Fold Symmetry

One can start with 5 kites (or 5 darts) arranged in a symmetric pattern and subdivide repeatedly. Taking the limit of this process, one can produce two distinct Penrose tilings with 5 fold symmetry. Figures 7 and 8 show the beginnings of these two tilings.

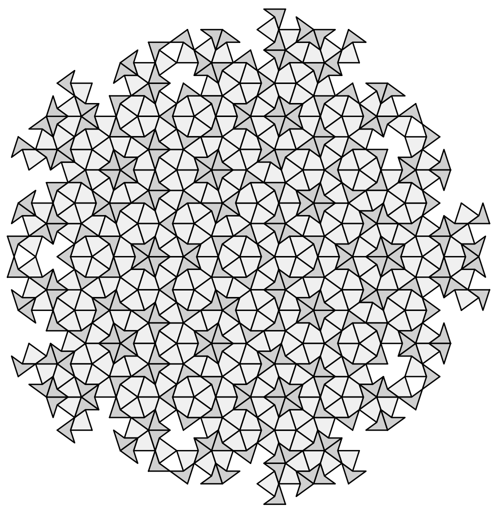

Figure 7

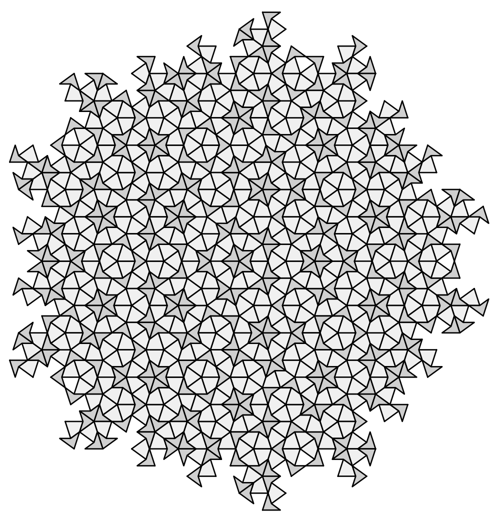

Figure 8

Let P_1 be the first of these tilings and let P_2 be the second. By construction P_1 is the subdivision of P_2 and P_2 is the subdivision of P_1.

Theorem 5.1 *Any Penrose tiling with* 5*-fold symmetry is either* P_1 *or* P_2*.*

Proof: Let Q be a Penrose tiling with 5-fold symmetry. Let B be some huge ball. We'll show that $Q \cap B$ agrees with one of $P_1 \cap B$ or $P_2 \cap B$. Since the size of B is arbitrary, this suffices to show that Q agrees with one of P_1 or P_2 on the whole plane.

Given any Penrose tiling X, let X^{-n} denote the Penrose tiling such that X is the nth subdivision of X. This notation is exactly like what we had in the previous section. For n large enough $Q^{-n} \cap B$ is contained in at most 5 tiles. Moreover, $Q^{-n} \cap B$ has 5-fold symmetry because Q does. But then $Q^{-n} \cap B$ is either 5 darts arranged in a symmetric pattern, or 5 kites arranged in a symmetric pattern. But then $Q \cap B$ is obtained by repeatedly subdividing either 5 kites or the 5 darts. But this is how we produced pieces of P_1 or P_2. This shows that (up to rotation) $Q \cap B$ equals either $P_1 \cap B$ or $P_2 \cap B$. ♠

Say that a *pentagonal seed* is a group of 5 kites arranged around a vertex. Both the tilings P_1 and P_2 above are produced from repeated subdivision of a pentagonal seed.

Theorem 5.2 *There is a number D_0 with the following property. If P is any Penrose tiling and x is any point of the plane, then there is some pentagonal seed of P within D_0 units of x.*

Proof: Looking at Figures 4-6, we see that the subdivision of a single kite leads to a pentagonal seed after 2 iterations. We can write $P = X''$, where X is another Penrose tiling. In other words, P is the second subdivision of X. Note that X'' has lots of pentagonal seeds, because every kite of X gives rise to a seed in X''. But $X'' = P$. ♠

Say that an n-seed is the result of subdividing a Pentagonal seed n times. For large n, an n-seed is a huge portion of one of the 5-fold symmetric Penrose tilings. The following result says that any Penrose tiling has huge finite regions that have 5-fold rotational symmetry, even though the whole tiling might not have 5-fold rotational symmetry.

Theorem 5.3 *There is a number D_n with the following property. If P is any Penrose tiling and x is any point of the plane, then x is within D_n units of some n-seed of P.*

Proof: This is the same proof as for the previous result, except that we write $P = X^{n+2}$, the $(n+2)$nd subdivision of X. Then P is the nth subdivision of X''. We know that X'' has pentagonal seeds all over the place, so X^{n+2} has n-seeds all over the place. ♠