# Random Walks and Electric Networks

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Most of the things in these notes can be found in the book *Random Walks and Electric Networks*, by Peter Doyle and Laurie Snell. I highly recommend this excellent book. These notes cover some of the main points in the book, but they do not always do things as they are in the book.

## 1 Random Walks on Graphs

Let $G$ be a graph in which every vertex has finite degree. We usually take $G$ to be a finite graph, but sometimes we will consider countable graphs. Suppose that we have given a linear ordering to the edges incident to each vertex of $G$. A *random walk* on $G$, starting from the vertex $v \in G$, is a sequence of fair coin flips $b_1, b_2, b_3, ...$ where the $(j)$th coin has as many aides as the degree of the vertex $v_j$. The value of $b_1$ selects the vertex $v_2$ adjacent to $v$, the value of $b_2$ selects the vertex $v_3$ adjacent to $v_2$, and so on. Here we have set $v_1 = v$. One funny thing about this process is that the number of sides of the coin can vary from vertex to vertex. If you prefer, one can consider random walks on regular graphs, and then one can use the same coin all the time.

Suppose that $A$ and $B$ are two totally distinct subsets of vertices. We define $P(v, A, B)$ to be the probability that a random walk starting from $v$ reaches $A$ before it reaches $B$. Some of you might be satisfied that this notion of probability makes intuitive sense. In this case, just skip the next section. Otherwise, you can read a quick sketch of how this probability is defined in terms of measure theory.

In any case, we are really only going to use a few basic properties of the above function. Suppose that $w_1, ..., w_k$ are the vertices incident to $v$ and

$v \in G - A - B$. Then the basic property is

$$P(v, A, B) = \frac{1}{k} \sum_{i=1}^{k} P(w_i, A, B). \quad (1)$$

In other words, we have an equal chance of going from $v$ to each $w_i$, and then we can compute the probability of hitting $B$ before $A$ and just average these probabilities.

# 2 Measure Theoretic Aside

Here is the way the probability $P(v, A, B)$ is treated from the standpoint of measure theory. Again, if you are happy with the definition already, just skip this part of the notes.

In general, the set of all possible coin flips is the subset $S(G, v)$ of infinite allowable integer sequences. A sequence is *allowable* if, for all $j$, the $j$th digit $b_j$ does not exceed the degree of the vertex $v_j$ selected by the previous terms $b_1, ..., b_{j-1}$. For regular graphs of degree $d$, the set $S(G, v_0)$ is just the set of all infinite sequences involving $d$ digits.

We let

$$S = S(G, v) \quad (2)$$

Supposing that we have chosen initial allowable sequence $\beta = (b_1, ..., b_n)$, the *cyclinder set* $C_\beta$ is the set of all allowable infinite sequences which have start with $\beta$. The probability that a finite random walk of length $n$ starting at $v$ will produce $\beta$ is

$$|C_\beta| = \frac{1}{d_1 ... d_n}, \quad (3)$$

where $d_j$ is the degree of $v_j$.

In general, one defines the *outer measure* of a subset $E \subset S$ as follows.

$$\mu^*(E) = \inf_{\mathcal{C}} \sum_{C \in \mathcal{C}} |C|. \quad (4)$$

Here $\mathcal{C}$ is a covering of $S$ by cylinder sets and $|C|$ denotes the probability that $C$ occurs, as above. What we are doing is taking the infimum over all possible coverings.

A subset $E$ is called *measurable* if

$$\mu^*(X - E) + \mu^*(X \cap E) = \mu^*(X) \quad (5)$$

for all other subsets $X \subset S$. In this case, we define $\mu(E) = \mu^*(E)$. This definition looks insane, because it is something we would have to test for *all* other subsets. However, one can check several basic properties:

- Cylinder sets themselves are measurable. This is pretty easy. If we have a covering of $X$ we get coverings of $X \cap E$ just by intersecting the cover with $E$ and we get a covering of $X - E$ by intersecting our covering with $X - E$, which is a finite union of cylinder sets.

- If $E$ is measurable so is $S - E$. This just follows straight from the definition.

- The countable union of measurable sets is again measurable. This is a bit more work, but not too bad.

A set is called a *Borel set* if it is obtained by starting with cylinder sets and taking complements and countable unions finitely many times. From the three properties above, Borel sets are measurable.

The function $\mu$ is called a *Borel measure*. The pair $(S, \mu)$ is called a *probability space*. Subsets of $S$ are often called *events*. When $E$ is a Borel event, $\mu(E)$ is the probability that $E$ occurs. The measure $\mu$ is *countably additive*: If $E_1, E_2, E_3, ...$ is any countable collection of pairwise disjoint measurable subsets of $S$, then

$$\mu(\bigcup E_j) = \sum \mu(E_j). \tag{6}$$

The set $E(v, A, B)$ of random walks which start at $v$ and hit $A$ before $B$ is one of these Borel sets. We define

$$P(v, A, B) = \mu(E(v, A, B).. \tag{7}$$

Let us sketch the derivation of Equation 1. Let $E = E(v, A, B)$ and also define $E_j = E(w_j, A, B)$. Note that $E_j$ is a subset of the space $S(w_j, A, B)$. Finally let $E'_j \subset S$ denote the set of random walks which first go to $w_j$ and then lie in $E_j$. We have

$$\mu(E_j) = (1/k)\ \mu_j(E'_j) \tag{8}$$

Here $\mu_j$ is the measure on $S(G, w_j)$. Also, from Equation 6 (in the finite case) we have $\mu(E) = \sum_j \mu(E_j)$. Putting these two facts together gives Equation 1.

## 3 Harmonic Functions

Let $f : G \to \boldsymbol{R}$ be a function. The function $f$ is called *harmonic* at a vertex $v$ if

$$f(v) = \frac{1}{k}\sum_{i=1}^{k} f(w_i). \tag{9}$$

Here $k$ is the degree of $v$ and $w_1, ..., w_k$ are the vertices incident to $v$. Notice the similarity between this equation and Equation 1.

Suppose that the disjoint vertex sets $A$ and $B$ have been fixed. Suppose that $f$ is some function on $A \cup B$. We call the function $F : G \to \boldsymbol{R}$ a *harmonic extension* of $f$ if $F = f$ on $A \cup B$ and $F$ is harmonic on $G - A - B$. To be clear, we are talking about functions defined on the vertex set of $G$. We will show that each $f$ has a unique harmonic extension.

**Lemma 3.1** *If $F_1$ and $F_2$ are two harmonic extensions of $f$ then $F_1 = F_2$.*

**Proof:** Let $g = F_1 - F_2$. Note that $g$ is a harmonic extension of the 0 function on $A \cup B$. There must be some vertex $v$ where $g$ achieves its maximum. But $g(v)$ is the average of the values of $g$ at the vertices incident to $v$. This is only possible if $g$ also takes its max at all the vertices incident to $v$. Continuing outward from $v$, we see that $g$ must take its max everywhere. But then this common value must be 0. ♠

**Lemma 3.2** *Every function $f$ on $A \cup B$ has a harmonic extension.*

**Proof:** Define the *graph laplacian* to be

$$\Delta_f g(v) = v - \frac{1}{k}\sum g^*(w_i). \tag{10}$$

As usual $k$ is the degree of $v$ and $w_1, ..., w_k$ are the vertices incident to $v$. The function $g^*(w_j)$ is $g(w_j)$ if $w_j \in G - A - B$ and $f(w_j)$ otherwise.

Let $V$ denote the vector space of functions on $G - A - B$. The map $\Delta_f$ is an affine transformation from $V$ into $V$. (It is matrix multiplication followed by translation which depends on the function $f$.)

Suppose that $\Delta_f(F_1) = \Delta_f(F_2)$. Then, setting $g = F_1 - F_2$ we have

$$\Delta g = \Delta_f(F_1) - \Delta_f(D_2).$$

Here $\Delta$ is the Laplacian defined for the 0-function on $A \cup B$. The same argument as in the previous result shows that $g = 0$ everywhere. This shows that $\Delta$ is one to one. But then $\Delta$ must be onto. In particular, the 0 function on $G - A - B$ is in the image of $\Delta_f$. That is, there exists $F \in V$ such that $\Delta_f(F)$ is the 0 function. By definition $F$ is a harmonic extension of $F$. ♠

Now we know that every function on $A \cup B$ has a unique harmonic extension. Suppose that we set $f = 1$ on $A$ and $f = 0$ on $B$. Let's define

$$F(v) = P(v, A, B). \tag{11}$$

Note that $F = f$ on $A \cup B$. Equation 1 tells us that $F$ is harmonic for all $v \in G - A - B$. In other words, the probability function $v \to P(v, A, B)$ we have been considering in previous sections is the unique harmonic extension of the function which is 1 on $A$ and 0 on $B$.

# 4 Electric Flow

We keep the same notation as above. We imagine $G$ as an electric network made of wires which all have the same resistance. We think of $A$ as a set of *sources* and $B$ as a set of *sinks*. Imagine using some kind of battery to keep all the vertices of $A$ at voltage 0 and all the vertices of $B$ at voltage 1. Electric current will then flow through the network. The electric current is determined as follows.

1. Let $F(v) = P(v, A, B)$ be the harmonic function defined above.
2. The current $j(v, w)$ that flows from vertex $v$ to vertex $w$ if $F(w) - F(v)$.

One can take this as the definition of electric current, but this definition obeys the usual laws one learns about in physics. First,

$$\Delta V = jR,$$

where $R$ is the resistance and $\Delta V$ denotes the change in voltage. Here we are setting $R = 1$ for every edge. Second, the total flow of current into any vertex in $G - A - B$ is the same as the total flow out. These two laws characterize the electric flow.

## 5 Two Examples

Before going further, let's work out two examples.

**Paths:** Suppose that $P$ is the path of length $n$. Let $v_0, ..., v_n$ be the vertices of $G$. Let $A = \{v_n\}$ and $B = \{v_0\}$. We define $F(v_j) = j/n$. Note that $F = 1$ on $A$ and $F = 0$ on $B$. Also, clearly $F$ is harmonic. In this case, the electric current is just $1/n$ on every edge, flowing from $A$ to $B$.

This analysis tells us something interesting about probabilities. If we start at $v_j$ the probability of hitting $A$ before $B$ is $j/n$. In particular, if we start at $v_1$ then the probability of hitting $A$ before $B$ is $1/n$. Hence, the probability of hitting $B$ before $A$ is $(n-1)/n$. When $n$ is large, this is almost a certainty. So, if we start our random walk at $B$, the probability of *returning* to $B$ before hitting $A$ is $(n-1)/n$: We move to $v_1$ with probability 1 because this is the only option, and then the chance of returning to $B$ before hitting $A$ is $(n-1)/n$. This suggests that that on the infinite line, the probability of returning to any given vertex in a random walk is 1. We will have more to say about this later.

**Infinite Binary Tree** Let $G$ be an infinite direcetd binary tree, with an initial node $v$. The tree has two edges pointing out of each note. Hence $v$ has degree 2 and all other vertices have degree 3. Suppose that $v$ points t $v_0$ and $v_1$, and $v_0$ points to $v_{00}$ and $v_{01}$, and $v_1$ points to $v_{10}$ and $v_{11}$ and so on. We let $A = \{v\}$ and $B = \emptyset$. We let $f(v) = 1$ and in general

$$f(v_s) = \frac{1}{2^k}, \qquad k = \text{length}(s).$$

Here $v_s$ is the vertex associated to the binary string $s$. The function $f$ is harmonic on $G - A$! For instance,

$$f(v_0) = \frac{1}{2} = \frac{f(v_0) + f(v_{00}) + f(v_{01})}{3} = \frac{1}{3}\Big(1 + \frac{1}{4} + \frac{1}{4}\Big).$$

For the remaining vertices, the calculation is similar. One can easily workout the electric current for this example.

In this example we think of $B$ as the set of ends of the tree. Note that $f(w) \to 0$ as $w$ moves away from $v$. In any case, the probability that a random walk starting at $v_s$ returns to $A$ is $1/2^k$ where $k$ is the length of $s$. In particular, a random walk that starts at the root vertex $v$ has a 50 percent chance of returning to $v$.

# 6 Dealing with Multiple Sources and Sinks

We are interested in triples $(G, A, B)$ as above. We will have a harmonic function $f : G \to \boldsymbol{R}$ which is constant on $A$ and constant on $B$. The various equations come out easier if we assume that $A$ and $B$ are each single points. We can always reduce to this case with a trick. Let $G^*$ be the new graph in which both $A$ and $B$ are collapsed to single points $a^*$ and $b^*$. Given a harmonic function $F$ as above, we define $F^*$ on $G^*$ so that $F^*(a^*) = F|_A$ and $F^*(b^*) = F|_B$ and otherwise $F = F^*$. This function is harmonic on $G^*$. We can reverse the process if we like. Starting with $F^*$ we get $F$. So, we always consider the case $(G, a, b)$ where $a$ and $b$ are vertices of $G$.

# 7 Effective Resistance

$F : G \to \boldsymbol{R}$ be the harmonic function such that $f(a) = 1$ and $F(b) = 0$. We define $R(G, a, b) = 1/i_a$, where $i_a$ is the total current flowing out of vertices of $A$. That is,

$$i_a = \sum_{x \leftrightarrow a} (F(a) - F(x)). \tag{12}$$

Equivalently, let $V : G \to \boldsymbol{R}$ be the harmonic function so that $V(b) = 0$ and $i_a = 1$. Then $R(G, a, b) = V(a)$. The function $V$ will be our constant companion. We call $V$ the *voltage function*. The current is always taken with respect to $V$.

**Lemma 7.1** *$i_a + i_b = 0$.*

**Proof:** This is a consequence of Kirkhoff's law. Here's the proof. Define $i_{xy} = V(x) - V(y)$ when $x$ and $y$ are connected by an edge and otherwise $i_{xy} = 0$. Setting

$$i_x = \sum_y i_{xy}, \tag{13}$$

, we have $i_x = 0$ unless $x = a$ or $x = b$. This is Kirkhoff's law, and it follows from the fact that $V$ is harmonic on $G - a - b$. But then

$$i_a + i_b = \sum_x i_x = \sum_x \sum_y i_{xy} = 0 \tag{14}$$

The last sum is 0 because $i_{xy} = -i_{yx}$ and we are summing over all pairs. ♠

Now we come to the main point of the section. Define the *energy dissipation*

$$E(G,a,b) = \frac{1}{2}\sum_{xy} i_{xy}^2. \tag{15}$$

The factor of 1/2 is added because $i_{xy}^2 = i_{yx}^2$ and both terms appear in the sum.

**Theorem 7.2** *$R(G,a,b) = E(G,a,b)$.*

**Proof:** Imagine that $\phi : G \to \boldsymbol{R}$ is any function. We will establish the identity

$$i_a(\phi(a) - \phi(b)) = \frac{1}{2}\sum_{xy} i_{xy}(\phi(x) - \phi(y)). \tag{16}$$

The current here is defined in terms of $V$. The function $\phi$ is just an arbitrary function. Note, however, that when we take $\phi = V$ the left hand side of te equation is $R(G,a,b)$ and the right side if $E(G,a,b)$.

The reason we consider the more general inequality is that both the left hand side and the right hand side are linear functions of $\phi$. So, to prove this equality it suffices to prove it on a basis in the vector space of such functions. Suppose that $\phi(a) = 1$ and $\phi(x) = 0$ otherwise. Then the two sides are equal just by definition. Suppose that $\phi(x) = 1$ for some $x \in G - a - b$. Then the left side is obviously 0, and the right side is given by

$$\sum_{i=1}^{k} i_{xy_k},$$

and this vanishes by Kirkhoff's law. Here $y_1, ..., y_k$ are the vertices incident to $x$. Finally, consider the case when $\phi$ is identically 1. Then both sides vanish. We've established the identity on a basis, so we're done. ♠

**Remarks:**
(i) We could have ended the proof differently. For our last function we could have used the function which is 1 on $b$ and 0 elsewhere. The result would then have follows from the fact that $i_a + i_b = 0$ and some algebra.
(ii) I guess that the physical intuition behind the result is that it takes a lot of effort to push one unit of current through a highly resistant material, and this effort is measured in terms of the heat dissipation.

# 8 A Variational Principle

Let $(G, a, b)$ and $V$ and $i_{xy}$ be as above. Here we explain a variational principle that characterizes the current $i_{ix}$ as the one which minimizes the energy dissipation. Say that a *nice flow* is a function $j : G \times G \to \boldsymbol{R}$ such that

1. $j_{xy} = 0$ if $x$ and $y$ do not share an edge.
2. $j_{xy} = -j_{yx}$.
3. If $x \in G - a - b$ then $j_x = 0$. That is, the amount of current flowing into $x$ is the same as the amount flowing out. Here $j_x = \sum j_{xy}$.

If, additionally, $j_a = 1$, we call $j_a$ a *unit flow*.

**Example:** Notice that if we take $j = i$ then the axioms are satisfied. But, there are plenty of examples of unit flows other than the electric flow. Here is an example. Suppose that $G'$ is a graph obtained by adding some edges to $G$. Then the current $i$ defined relative to $(G, a, b)$ gives a unit flow relative to $(G', a, b)$ but this unit flow on $G'$ might not be the current defined relative to $(G', a, b)$. The extra wires might change the current.

Define

$$E(j) = \frac{1}{2} \sum_{xy} j_{xy}^2. \tag{17}$$

We already know that $R(G, a, b) = E(i)$. Here is the main result:

**Theorem 8.1** *$E(i) \leq E(j)$. In other words, $R(G, a, b)$ is the min of all $E(j)$ taken over all unit flows $j$.*

**Proof:** The crucial fact is that Equation 16 holds for any nice flow. All we used in the proof is Axiom 3 above. Also, note that the nice flows form a vector space. Let $V$ be this vector space. There is a canonical inner product on $V$, namely

$$\langle j, k \rangle = \frac{1}{2} \sum_{xy} j_{xy} k_{xy} \tag{18}$$

With this formalism, we have $E(j) = \langle j, j \rangle$. Suppose that $k$ is a nice flow with $k_a = 0$. Then

$$\langle k, i \rangle = \sum_{x,y} k_{xy} i_{xy} = \sum_{x,y} k_{xy}(V(x) - V(y)) =_* k_a(V(a) - V(b) = 0. \tag{19}$$

The starred equality is Equation 16 applied to the nice flow $k$. This equation crucially uses the fact that $i$ is the electric flow.

Now let $j$ be any unit flow. We can write $j = i + k$ where $k \in V$ satisfies $k_a = 0$. But then

$$E(j) = \langle i+k, i+k \rangle = E(i) + E(k) \geq E(i).$$

Here have used the fact that $\langle i, k \rangle = 0$. ♠

# 9 Rayleigh's Theorem

Now we can prove the main technical result.

**Theorem 9.1** *Let $G'$ be a graph obtained from $G$ by adding some edges. Then $R(G', a, b) \leq R(G, a, b)$.*

**Proof:** We have $R(G, a, b) = E(G, a, b)$ and $R(G', a, b) = E(G', a, b)$. So, it suffices to prove that $E(G", a, b) \leq E(G, a, b)$. Let $i$ be the electric flow on $G$ relative to $a, b$. Let $i'$ be the electric flow on $G'$ relative to $a, b$. Note that $i$ is also a unit flow on $G'$. Hence

$$E(G', a, b) = E(i') \leq E(i) = E(G, a, b)$$

Each of the steps is one of the two theorems we proved above. ♠

# 10 Variable Conductance

The same results as above, including Rayleigh's Theorem, go through if we have a graph with variable conductances.

In general, imagine that we specify a *conductance* $C_{xy}$ for each graph edge $x \leftrightarrow y$. This time we require $C_{xy} = C_{yx}$. The quantity $C_{xy}$ encodes the physical properties of the material. If the edge was made of copper, $C_{xy}$ would be large. If the edge was made of thread, $C_{xy}$. The resistance of the edge is $1/C_{xy}$. Assuming that all the conductances have been specified, one defines a function $F : G \to \boldsymbol{R}$ to be harmonic at $x \in G - a - b$ if

$$F(x) = \frac{\sum_{i=1}^{k} C_i F(y_i)}{\sum_{i=1}^{k} C_i} \tag{20}$$

Here $y_1, ..., y_k$ are the vertices incident to $x$ and we have set $C_i = C_{xy_i}$ for notational convenience.

Once we make this change, all the results above go through and they have the same proofs. Likewise, all the results go through if we allow multiple edges connecting a pair of vertices. Having multiple edges connecting a pair of vertices is really just the same as having a single edge but changing its conductance. For instance, if two vertices are connected by an edge of conductance $a$ and an edge of conductance $b$, then this is equivalent to connecting these vertices by a single edge of conductance $a + b$.

Once we think about things in terms of multiple edges, we can think of the notion of *short circuiting* the graph $G$. Suppose that $X \subset G$ is some subset of vertices disjoint from $a$ and $b$. Let $G_X$ denote the new graph obtained by collapsing all the vertices of $X$ to a single point.

**Corollary 10.1** $R(G_X, a, b) \leq R(G, a, b)$.

**Proof:** By induction on the number of vertices of $X$, it suffices to consider the case when $X$ just consists of 2 vertices $c$ and $d$. Imagine that we add another edge from $c$ to $d$ having conductance $N$. Rayleigh's theorem tells us that the resistance of the new graph does not increase. Equation 20 makes sense even when some $C_i$ is set to $\infty$. In this case, Equation 20 just says that $F(x) = F(y_i)$ if $F$ is harmonic. So, taking $N = \infty$, and setting all the other conductances to 1, we get a new graph $G'$ such that any harmonic function on $G'$ must take on the same values at $c$ and $d$. But then any harmonic function on $G'$ gives a harmonic function on $G_X$ in the obvious way, and vice versa. Hence, $R(G_X, a, b) = R(G', a, b) \leq R(G, a, b)$. ♠

# 11 Graphs in Parallel

Here is a short discussion about graphs which appear in parallel. Suppose that $c \in G - a - b$ is a cut vertex of $G$. Let $G_a$ be the lobe of $G$ which contains $a$ and let $G_b$ be the lobe of $G$ which contains $b$.

**Lemma 11.1 (Parallel)** $R(G, a, b) = R(G_a, a, c) + R(G_b, c, b)$.

**Proof:** Let $V$ be the voltage function for $G$. Let $V'$ be the restriction of $V - V(c)$ to $G_a$. By construction $V'$ is harmonic on $V_a$ and $i'_a = 1$. Hence

$$R(G_a, a, c) = V(a) - V(c).$$

A similar argument shows that

$$R(G_b, b, c) = V(c) - V(b) = V(c).$$

The point of this second equation is that $i_b = -i_a$, so the function $-(V - V(c))$ is the voltage function for $(G_b, b, c)$. Adding up these equations gives us the desired result. ♠

# 12 Polya's Recurrence Theorem

Let $G_\infty$ denote the usual infinite graphs of edges of the unit square tiling of the plane. Polya's Recurrence Theorem says that a random walk in the pla will return with probability 1 to its starting point. Here I'll give a proof. By symmetry, it suffices to consider the case when the starting point is the origin.

Let $G_n$ be be the subgraph of $G_\infty$ consisting of the edges of the $(2n) \times (2n)$ square grid centered at the origin. Let $A = \{0, 0\}$ and let $B_n$ denote the outer cycle of $G_{2n}$. Even though $B_n$ is not a single point, the collapsing trick above makes Rayleigh's Theorem hold even for triples $(G_n, A, B_n)$. Let $P_n$ denote the probability that a random walk starting at $A$ will return to $A$ before hitting $B_n$. From the analysis above,

$$P_n = \frac{R_n - 1}{R_n}, \qquad R_n = R(G_n, A, B_n). \tag{21}$$

**Theorem 12.1** $\lim_{n\to\infty} R(G_n, A, B_n) \to \infty$.

**Proof:** The cycles $B_1, B_2, ..., B_n$ all exist in $G_n$, and they are disjoint. Let $G'_n$ denote the graph obtained by collapsing each of these cycles to a point. The graph $G'_n$ looks like the right side of Figure 1. Figure 1 shows the case $n = 2$. Here we have replaced the multiple edges of conductance 1 by an edge whose conductance is just the number of these original edges.

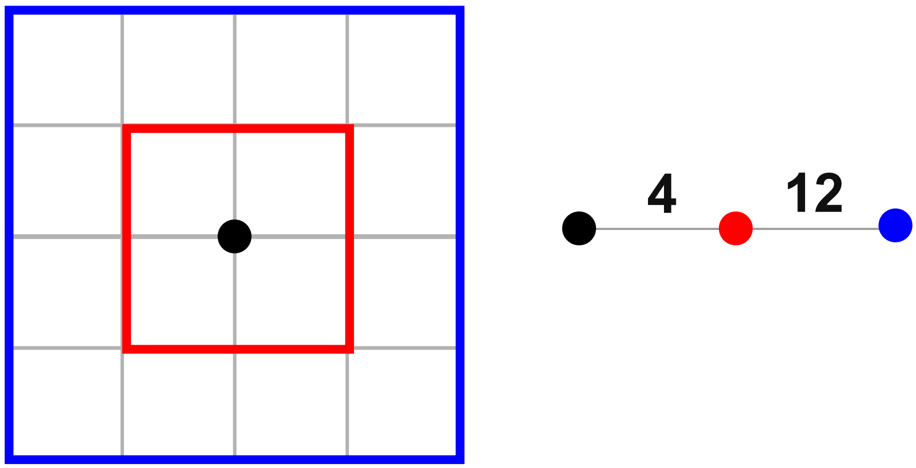


**Figure 1:** Collapsing the cycles

The collapse of $G_n$ is just a path of length $n$ whose conductances are $4 \times 1, 4 \times 3, 4 \times 5, \ldots$. From repeated applications of the Parallel Lemma,

$$R(G_n, A, B_n) \geq 4\left(\frac{1}{1} + \frac{1}{3} + \ldots + \frac{1}{2n+1}\right) \tag{22}$$

This series is closely related to the harmonic series, and is easily seen to diverge. ♠

As an immediate corollary, we see that $R_n \to 1$. If a random walk starting at the origin had probability $\epsilon > 0$ of never returning to the origin, then we would have $R(G_n, A, B_n) \geq \epsilon$ for all $n$. But this contradicts what we have just proved. This completes the proof of Polya's Theorem.

# 13 Polya's Transience Theorem

Now let $G_\infty$ denote the infinite cubical graph in $\boldsymbol{R}^3$. Polya's Transience Theorem says that a random walk starting at any point of $G_\infty$ has a nonzero probability of never returning to that point. Let $G_n$ be defined as above, with *cubes* replacing *squares*. A similar kind of argument as in the recurrence case shows that Polya's Transience Theorem is equivalent to the statement that $R(G_n, A, B_n) < N$ for some $N$ that does not depend on $n$. The idea of the proof is to find a tree $T_n \subset G_n$ such that

$$R(T_n, A, B_n \cap T_n) < N.$$

Now we're going to construct the tree.

**Pixellation:** Given any line segment in $\boldsymbol{R}^3$ which connects two points in $\boldsymbol{Z}^3$, there is a nearby path in $G_\infty$ which stays within one unit of the line segment. We just take the union of unit cubes containing the segment and we take a path on the boundary. The length of this path is comparable to the length of the line segment. The two quantities differ by at most a factor of 2. We choose such a path for every line segment and call this path the *pixellation* of the path. If $S$ is the segment, we let $S^*$ be the pixellation of $S$.

Call segments $S_1$ and $S_2$ *remote* if no point on $S_1$ is within 3 units of a point of a point on $S_2$. If $S_1$ and $S_2$ are remote then $S_1^*$ and $S_2^*$ are disjoint paths in our graph.

**Big Sisters and Entourages** Let's think of this in social terms. Given any point $p \in \boldsymbol{Z}^3$, let $S_p$ be the path joining $p$ to $2p$. We think of $2p$ as the *big sister* of $p$. If $p$ and $q$ are at least 3 units apart then $S_p^*$ and $S_q^*$ are disjoint paths.

Given any point $p_0 \in \boldsymbol{Z}^3$, we define the *friends* of $p$ to be the two points $P_1 = p_0 + (3,0,0)$ and $p_2 = p_0 + (0,3,0)$. We call the triple $(p_0, p_1, p_2)$ an *entourage*. If $p_0$ and $q_0$ are at least 6 units apart, then the corresponding entourages are at least 3 units apart.

**Main Construction:** Join $(0,0,0)$ to $p_0 = (0,0,1)$ by an edge. Form the entourage $(p_0, p_1, p_2)$. Join $p_0$ to its friends by straight line line segments. How join each of these points to their big sisters by the pixellated paths. Let these big sisters form their entourages. Join each big sister to her two friends, and then join all 9 points to *their* big sisters. And so on.

This construction produces an embedded directed tree $T_n$ whose initial node is $(0,0,0)$ and whose outgoing edges always point from vertices to their big brothers. Except for the initial node and the terminal nodes, every other node has one incoming edge and 3 outgoing edges. There is 1 outgoing edge of length 1, and then 3 outgoing edges of length $O(2)$ and 9 outgoing edges of length $O(4)$ and in general $3^n$ outgoing edges of length $O(2^n)$. Here the notation means that the length of these edges is at most, say, $4 \times 2^n$.

This is $T_n$. If necessary lengthen some of the paths of $T_n$ by adding extra vertices on some of the edges so that at the $n$th stage, all edges have length $4 \times 2^n$. Rayleigh's theorem says that this process does not decrease the resistance. The new tree $T_n'$ is no longer a subgraph of $G_n$ but we don't care. What we know is that

$$R(T_n, A, B_n \cap T_n) \leq R(T_n', A, B_n \cap T_n).$$

We just have to bound this latter quantity.

Why did we add these edges. Well, by symmetry the voltage function on $T_n'$ takes the same value on all vertices that are the same distance from the initial node! So, we can collapse all the vertices at the same distance from the initial node and we get a graph with *the same resistance*. The collapsed graph is just a path (of length about $\log_2(n)$) whose edges have conductance

$$\frac{3^k}{4 \times 2^k}, \qquad k = 1, 2, 3, ...$$

The corresponding resisistance is at most $4 \times (2/3)^k$. Since the sum

$$N = 4 \times \sum_k^{\infty} (2/3)^k$$

is finite the we have $R(T_n', A, B_n \cap T_n) < N$ for all $n$. This completes the proof.