1 The Siegel Domain

We equip $C^{2,1}$ with the Hermitian form
\[ \langle Z, W \rangle = Z_1 \overline{W}_3 + Z_2 \overline{W}_2 + Z_3 \overline{W}_1. \]

$CH^2$ is defined as the projectivization $[N_-] \subset CP^2$, where
\[ N_- = \{ V | \langle V, V \rangle < 0 \}. \]

This model of $CH^2$ is sometimes called the Siegel domain. The ideal boundary of $CH^2$ is the set $[N_0] \subset CP^2$, where
\[ N_0 = \{ V | \langle V, V \rangle = 0 \}. \]

There is an inclusion $C^2 \to CP^2$ given by the equations
\[ (z, w) \to [z : w : 1]. \]

In this way, we identify $C^2$ with a subset of $CP^2$. We have
\[ [N_0] \cap C^2 = Z, \quad Z = \{ (z, w) | \Re(z) = -|w|^2/2 \}. \]

In fact, $[N_0]$ is the one-point compactification of $Z$. If we define
\[ \infty = [1 : 0 : 0], \]
then
\[ [N_0] = Z \cup \infty. \]

The set $Z$ is the boundary of the Siegel domain.
2 The Heisenberg Group

The Heisenberg Group is the space $C \times R$, equipped with the group law

$$(z_1, t_1) \ast (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \Im(\overline{z}_1 z_2)).$$

That weird symbol denotes “imaginary part”. We’ll denote the Heisenberg group by $H$. The space $H$ is closely related to the complex hyperbolic plane.

There is a natural map $Z \to H$, given by

$$(z, w) \to (w, -\Im(z)).$$

The inverse map is given by

$$(w, t) \to (-it - |w|^2/2, w).$$

Let’s work out the symmetries of $CH^2$ in terms of their action on $Z$ and (using the map above) $H$.

Rotation: Let $u$ be a unit complex number. The map $(z, w) \to (z, uw)$ is clearly a symmetry of $Z$. The corresponding map on $H$ is given by

$$(w, t) \to (uw, t).$$

Geometrically, this rotates around the vertical axis in $H$. One can check that this map is an automorphism of $H$, with respect to the above group law.

Vertical Translation: Let $s \in R$. The map $(z, w) \to (z - is, w)$ is a symmetry of $Z$. The corresponding map on $H$ is given by

$$(w, t) \to (0, s) \ast (w, t).$$

So, we get left multiplication by the element $(0, s)$.

Dilation: Let $s > 0$ be real. The map $(z, w) \to (s^2z, sw)$ is a symmetry of $Z$. The corresponding map on $H$ is given by

$$(w, t) \to (sw, s^2t).$$

This is also an automorphism of $H$. 

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Translation: Let $s \in \mathbb{R}$. The point $\left(-\frac{(y + s)^2}{2}, y + s\right)$ lies on $\mathcal{Z} \cap \mathbb{R}^2$ for all $s$. Expanding this out and using the fact that $x = -\frac{y^2}{2}$, we see that the map

$$(x, y) \rightarrow (x - sy - s^2/2, y + s)$$

(14)

preserves $\mathcal{Z} \cap \mathbb{R}^2$. But then this same map preserves $\mathcal{Z}$ when we allow $x$ and $y$ to be complex variables. In short, the affine map

$$(z, w) \rightarrow (z - sw - s^2/2, w + s)$$

(15)

preserves $\mathcal{Z}$. The corresponding map on $\mathcal{H}$ is given by

$$(w, t) \rightarrow (s, 0) * (w, t).$$

(16)

Again, we see left multiplication on $\mathcal{H}$.

Inversion: The map $(a, b, c) \rightarrow (c, b, a)$ is an obvious symmetry of the Hermitian form we have introduced. The corresponding map on $\mathcal{Z} \cup \infty$ is given by $(z, w) \rightarrow (1/z, w/z)$. This map interchanges $(0, 0) \in \mathcal{Z}$ with $\infty$. It is the analog of inversion. The corresponding action on $\mathcal{H} \cup \infty$ is given by

$$I(w, t) = \left(\frac{-w}{|w|^2/2 + it}, \Im \left(\frac{1}{|w|^2/2 + it}\right)\right).$$

(17)

Fibration: There is a canonical map $\pi : \mathcal{H} \rightarrow \mathbb{C}$, given by

$$\pi(w, t) = w.$$

(18)

We do not consider inversion to be a symmetry of $\mathcal{H}$. We call the other maps considered above symmetries. Every symmetry $T$ of $\mathcal{H}$ acts on $\mathbb{C}$ via the rule

$$T^*(w) = \pi \circ T(w, s).$$

(19)

The definition turns out to be independent of the choice of $s$. The map $T \rightarrow T^*$ carries $T$ to a similarity of $\mathbb{C}$. We say that $T$ covers $T^*$.

The fibration is a nice way to study the structures in $\mathcal{H}$, because we can project them to $\mathbb{C}$ and then draw them. This is one of secrets of understanding the 4-dimensional space $\mathbb{C} \mathbb{H}^2$: First look at how things intersect the boundary, then map to $\mathcal{H}$, then project to $\mathbb{C}$.
3 Structures in the Heisenberg Group

3.1 The Contact Plane Field

The map $Z \rightarrow H$ carries the complex line field tangent to $Z$ to a left-invariant plane field on $H$. We can figure out this plane field using the symmetries above. We call this plane field the contact plane field. We call the planes of the contact field the contact planes. The name comes from the connection to contact geometry.

Lemma 3.1 At $(0,0)$, the contact plane is $C \times \{0\}$.

Proof: Symmetry. The only plane through $(0,0)$ invariant under rotations is $C \times \{0\}$. ♠

Lemma 3.2 The contact plane field is the kernel of the 1-form

$$dz - (xdy - ydx).$$

Proof: At the point $(1,0)$, the plane is spanned by the vectors

$$(1,0), \quad (i,1).$$

In real notation, these vectors are $(1,0,0)$ and $(0,1,1)$. We get this from setting $s = 1$ in Equation 14 and considering the resulting action on $C \times \{0\}$. Note that both vectors are killed by the form $dz - dy$. So, the claim is true at $(1,0)$. But both the contact field and the kernel of the 1-form are invariant under rotations, dilations, and vertical translations. Hence, by symmetry, the claim is true everywhere. ♠

Let $\Pi$ be any contact plane. The restriction $\pi : \Pi \rightarrow C$ naturally gives $\Pi$ a complex structure – i.e. a canonical identification with $C$. We can say that a $\Pi$-circle is an ellipse $E \subset \Pi$ such that $\pi(E)$ is a circle. We will see this structure on $\Pi$ come up below when we discuss $C$-circles.
3.2 $\mathcal{C}$-circles

A $\mathcal{C}$-circle is the image in $\mathcal{H}$ if the boundary of a $\mathcal{C}$-slice of $CH^2$.

**Lemma 3.3** A straight line in $\mathcal{H}$ lies in a $\mathcal{C}$-circle iff the line is vertical.

**Proof:** The set

$$CH^2 \cap \{ [z_1 : 0 : z_3] \} \quad (21)$$

is a $\mathcal{C}$-slice. The corresponding set in $\mathcal{H}$ is the vertical line $0 \times \mathbb{R}$. It now follows from translation symmetry that any vertical line in $\mathcal{H}$ lies in a $\mathcal{C}$-circle.

Conversely, suppose that $L$ is a straight line lying in a $\mathcal{C}$-circle. Applying Heisenberg symmetries, we can assume that $L$ goes through the origin. But then $L$ corresponds to a $\mathcal{C}$-slice in $CH^2$ containing $(0,0)$ and $\infty$. Such a $\mathcal{C}$-slice is unique and must be the one we have already considered, namely $0 \times \mathbb{R}$. ♠

Call a curve in $\mathcal{H}$ non-straight if it is not a straight line.

**Lemma 3.4** A loop $X$ in $\mathcal{H}$ is a non-straight $\mathcal{C}$-circle if and only if $X$ is a $\Pi$-circle contained in the contact plane $\Pi$ through its own center of mass.

**Proof:** For the purposes of this lemma, call a loop good if it is a $\Pi$-circle contained in the contact plane $\Pi$ through its own center of mass. The one point compactification of the set

$$CH^2 \cap \{ (−1/2, w, 1) | |w| = 1 \} \quad (22)$$

is a $\mathcal{C}$-slice. The corresponding $\mathcal{C}$-circle in $\mathcal{H}$ is $S^1 \times \{0\}$, where $S^1$ is the unit circle in $\mathcal{C}$. Note that the center of mass of $X$ is the origin, and $X$ is contained in $\mathcal{C} \times \{0\}$ and $\pi(X) = S^1$. So, $S^1 \times \{0\}$ is good and also a $\mathcal{C}$-circle.

Using Heisenberg symmetries, we see that every good loop is a $\mathcal{C}$-circle. Conversely, we can move a $\mathcal{C}$-circle by Heisenberg symmetries so that it contains the two points $(\pm 1, 0)$. But then, by uniqueness of $\mathcal{C}$-slices, the loop we get must be the one we first considered. Moreover, the Heisenberg symmetries preserve goodness. Hence, all $\mathcal{C}$-circles are good. ♠
\section*{3.3 $R$-circles}

A $R$-circle is the image in $\mathcal{H}$ if the boundary of a $R$-slice of $\mathcal{C}H^2$.

We call a curve $\gamma \subset \mathcal{H}$ integral if it is always tangent to the contact plane field. The $C$-circles are not contact.

**Lemma 3.5** A straight line in $\mathcal{H}$ lies in an $R$-circle if and only if it is contact.

**Proof:** The set $\mathcal{C}H^2 \cap R^2$ is the image of the parabola $x < -y^2/2$. The ideal boundary is the parabola $x = -y^2/2$. The corresponding $R$-slice in $\mathcal{H}$ is the line $R \times \{0\}$. This line is integral.

The image of $R \times \{0\}$ under any Heisenberg symmetry is again an $R$-circle. It is not hard to show that any straight line integral to the plane field can be obtained this way. Hence, every integral straight line lies in an $R$-circle.

Conversely, any straight $R$-circle corresponds to an $R$-slice which can be moved to $\mathcal{C}H^2 \cap R^2$ by symmetries which fix $\infty$. Hence every straight $R$-circle is an integral straight line. ♠

We recall a definition from classical geometry. A **Lemniscate** is any curve which is similar to the one given, in polar coordinates, by $r^2 = \cos(2\theta)$. At the same time, a Lemniscate is what you get when you invert a square hyperbola $xy = a$.

**Lemma 3.6** A loop $X$ in $\mathcal{H}$ is non-straight $R$-circle if and only if $X$ is integral and $\pi(X)$ is a lemniscate.

**Proof:** Call $X$ **good** if $X$ is integral and projects to a lemniscate.

We apply the Heisenberg inversion $I$ from Equation 17 to the $R$-circle $\gamma = R \times \{1\}$. Parametrizing this $R$-circle by $(s, 1)$, we see that the first coordinate of $I(\gamma)$ is given by

$$\frac{-1}{s/2 + i/s} \quad \text{(23)}$$

But the curve $s \to s/2 + i/s$ lies on the hyperbola $xy = 1/2$. Hence the first coordinate of $I(\gamma)$ is a Lemniscate. Hence $I(\gamma)$ is both good and an $R$-circle.

The rest of the proof follows from the kind of symmetry used several times already. ♠
3.4 Bisectors and Spinal Spheres

Recall that a bisector in $\mathcal{C}H^2$ is the equidistant set between two points.

**Lemma 3.7** The set $X = (\mathbb{R} \times \mathbb{C}) \cap \mathcal{C}H^2$ is a bisector.

**Proof:** Consider the points

$$p_{\pm} = (-1 \pm i : 0 : 1) \in \mathcal{C}H^2. \quad (24)$$

Given $q = (s : w : 1) \in X$, we compute

$$\langle p_{\pm}, p_{\pm} \rangle = -2; \quad \langle q, p_{\pm} \rangle = (s - 1) \pm i. \quad (25)$$

Therefore, the quantity

$$\cosh(d(p_{\pm}, q)) = \frac{\langle p_{\pm}, q \rangle \langle q, p_{\pm} \rangle}{\langle p_{\pm}, p_{\pm} \rangle \langle q, q \rangle} = \frac{(s - 1)^2 + 1}{-2 \langle q, q \rangle} \quad (26)$$

is the same for both points $p_+$ and $p_-$. ♠

A spinal sphere is the ideal boundary of a bisector. We call a spinal sphere in $\mathcal{H}$ straight if it contains $\infty$.

**Lemma 3.8** A surface in $\mathcal{H}$ is a straight spinal sphere if and only if it is a contact plane.

**Proof:** The spinal sphere in $\mathcal{H}$ corresponding to the bisector we have just considered is $\mathcal{C} \times \{0\}$. This is certainly a contact plane. As above, the rest of the lemma follows from symmetry. ♠

The contact plane $\mathcal{C} \times \{0\}$ has a double (singular) foliation:

- The $\mathcal{C}$-circles $\{|z| = r\} \times \{0\}$ give one singular foliation.
- The straight $\mathbb{R}$-circles through the origin give the other one.

These give a natural polar coordinate system to the spinal sphere. By symmetry, these foliations exist on all the spinal spheres, not just the straight ones.

The non-straight spinal spheres are a bit messier to describe and I won’t do it. They are all smooth spheres with these two singular foliations.
4 The Carnot Metric

The Heisenberg symmetries discussed above all cover similarities of $C$. It turns out that there is a metric on $H$ such that all Heisenberg symmetries are similarities. The metric is defined as follows. On each contact plane $\Pi$, we put the Euclidean metric which makes $\pi : \Pi \to C$ an isometry. We then define the length of an integral curve to be the integral of its speed! This definition makes no sense for non-integral curves. We define the Carnot distance between two points to be the infimal length of an integral curve connecting them.

**Lemma 4.1** The Carnot distance from $(0,0)$ to $(0,2\pi r^2)$ is $2\pi r$.

**Proof:** Use the contact form $dz - (xdy + ydx)$, and Green’s Theorem, and the isoperimetric inequality. The best curve in this case is a helix which projects to the circle of radius $r$ centered at the origin. ♠

**Lemma 4.2** The Carnot distance from $(0,0)$ to $(r,0)$ is $r$.

**Proof:** The segment $[0,r] \times \{0\}$ is the best integral connector in this case. ♠

Notice that when $r$ is small, the distance from $(0,0)$ to $(0,\pi r^2)$ is huge in comparison to the Euclidean distance between the two points. Thus, the Carnot metric is not bi-lipschitz equivalent to the Euclidean metric (or any Riemannian metric) on $H$. Indeed, it turns out that the Hausdorff dimension of $H$, with respect to the Carnot metric, is 4-dimensional. This crazy-sounding statement is not so hard to prove. Using Lemmas 4.1 and 4.2 and symmetry, one can see that it takes about $O(\epsilon^{-4})$ Carnot-balls of diameter $\epsilon$ to fill up the unit Carnot ball.