Notes on Heisenberg Space

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1 The Siegel Domain

We equip $\boldsymbol{C}^{2,1}$ with the Hermitian form

$$\langle Z, W\rangle = Z_1\overline{W}_3 + Z_2\overline{W}_2 + Z_3\overline{W}_1. \tag{1}$$

$\boldsymbol{CH}^2$ is defined as the projectivization $[N_-] \subset \boldsymbol{CP}^2$, where

$$N_- = \{V|\ \langle V, V\rangle < 0\}. \tag{2}$$

This model of $\boldsymbol{CH}^2$ is sometimes called the *Siegel domain*.

The ideal boundary of $\boldsymbol{CH}^2$ is the set $[N_0] \subset \boldsymbol{CP}^2$, where

$$N_0 = \{V|\ \langle V, V\rangle = 0\}. \tag{3}$$

There is an inclusion $\boldsymbol{C}^2 \to \boldsymbol{CP}^2$ given by the equations

$$(z, w) \to [z : w : 1]. \tag{4}$$

In this way, we identify $\boldsymbol{C}^2$ with a subset of $\boldsymbol{CP}^2$. We have

$$[N_0] \cap \boldsymbol{C}^2 = \mathcal{Z}, \qquad \mathcal{Z} = \{(z, w)|\ \Re(z) = -|w|^2/2\} \tag{5}$$

In fact, $[N_0]$ is the one-point compactification of $\mathcal{Z}$. If we define

$$\infty = [1 : 0 : 0], \tag{6}$$

then

$$[N_0] = \mathcal{Z} \cup \infty. \tag{7}$$

The set $\mathcal{Z}$ is the boundary of the Siegel domain.

2 The Heisenberg Group

The *Heisenberg Group* is the space $\boldsymbol{C} \times \boldsymbol{R}$, equipped with the group law

$$(z_1, t_1) * (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \Im(\overline{z}_1 z_2)). \tag{8}$$

That weird symbol denotes "imaginary part". We'll denote the Heisenberg group by $\mathcal{H}$. The space $\mathcal{H}$ is closely related to the complex hyperbolic plane.

There is a natural map $\mathcal{Z} \to \mathcal{H}$, given by

$$(z, w) \to (w, -\Im(z)). \tag{9}$$

The inverse map is given by

$$(w, t) \to (-it - |w|^2/2, w). \tag{10}$$

Let's work out the symmetries of $\boldsymbol{CH}^2$ in terms of their action on $\mathcal{Z}$ and (using the map above) $\mathcal{H}$.

Rotation: Let u be a unit complex number. The map $(z, w) \to (z, uw)$ is clearly a symmetry of $\mathcal{Z}$. The correspinding map on $\mathcal{H}$ is given by

$$(w, t) \to (uw, t). \tag{11}$$

Geometrically, this rotates around the vertical axis in $\mathcal{H}$. One can check that this map is an automorphism of $\mathcal{H}$, with respect to the above group law.

Vertical Translation: Let $s \in \boldsymbol{R}$. The map $(z, w) \to (z - is, w)$ is a symmetry of $\mathcal{Z}$. The corresponding map on $\mathcal{H}$ is given by

$$(w, t) \to (0, s) * (w, t). \tag{12}$$

So, we get left multiplication by the element $(0, s)$.

Dilation: Let $s > 0$ be real. The map $(z, w) \to (s^2 z, sw)$ is a symmetry of $\mathcal{Z}$. The corresponding map on $\mathcal{H}$ is given by

$$(w, t) \to (sw, s^2 t). \tag{13}$$

This is also an automorphism of $\mathcal{H}$.

Translation: Let $s \in \boldsymbol{R}$. The point $(-(y+s)^2/2, y+s)$ lies on $\mathcal{Z} \cap \boldsymbol{R}^2$ for all s. Expanding this out and using the fact that $x = -y^2/2$, we see that the map

$$(x, y) \to (x - sy - s^2/2, y + s) \tag{14}$$

preserves $\mathcal{Z} \cap \boldsymbol{R}^2$. But then this same map preserves $\mathcal{Z}$ when we allow x and y to be complex variables. In short, the affine map

$$(z, w) \to (z - sw - s^2/2, w + s) \tag{15}$$

preserves $\mathcal{Z}$. The corresponding map on $\mathcal{H}$ is given by

$$(w, t) \to (s, 0) * (w, t). \tag{16}$$

Again, we see left multiplication on $\mathcal{H}$.

Inversion: The map $(a, b, c) \to (c, b, a)$ is an obvious symmetry of the Hermitian form we have introduced. The corresponding map on $\mathcal{Z} \cup \infty$ is given by $(z, w) \to (1/z, w/z)$. This map interchanges $(0, 0) \in \mathcal{Z}$ with ∞. It is the analog of inversion. The corresponding action on $\mathcal{H} \cup \infty$ is given by

$$I(w, t) = \left(\frac{-w}{|w|^2/2 + it}, \Im\left(\frac{1}{|w|^2/2 + it} \right) \right). \tag{17}$$

Fibration: There is a canonical map $\pi : \mathcal{H} \to \boldsymbol{C}$, given by

$$\pi(w, t) = w. \tag{18}$$

We do not consider inversion to be a symmetry of $\mathcal{H}$. We call the other maps considered above *symmetries*. Every symmetry T of $\mathcal{H}$ acts on $\boldsymbol{C}$ via the rule

$$T^*(w) = \pi \circ T(w, s). \tag{19}$$

The definition turns out to be independent of the choice of s. The map $T \to T^*$ carries T to a similarity of $\boldsymbol{C}$. We say that T *covers* T^*.

The fibration is a nice way to study the structures in $\mathcal{H}$, because we can project them to $\boldsymbol{C}$ and then draw them. This is one of secrets of understanding the 4-dimensional space $\boldsymbol{CH}^2$: First look at how things intersect the boundary, then map to $\mathcal{H}$, then project to $\boldsymbol{C}$.

3 Structures in the Heisenberg Group

3.1 The Contact Plane Field

The map $\mathcal{Z} \to \mathcal{H}$ carries the complex line field tangent to $\mathcal{Z}$ to a left-invariant plane field on $\mathcal{H}$. We can figure out this plane field using the symmetries above. We call this plane field the *contact plane field*. We call the planes of the contact field the *contact planes*. The name comes from the connection to contact geometry.

Lemma 3.1 *At* $(0,0)$*, the contact plane is* $\boldsymbol{C} \times \{0\}$.

Proof: Symmetry. The only plane through $(0,0)$ invariant under rotations is $\boldsymbol{C} \times \{0\}$. ♠

Lemma 3.2 *The contact plane field is the kernel of the* 1*-form*

$$dz - (xdy - ydx).$$

Proof: At the point $(1,0)$, the plane is spanned by the vectors

$$(1,0), \qquad (i,1). \tag{20}$$

In real notatation, these vectors are $(1,0,0)$ and $(0,1,1)$. We get this from setting $s=1$ in Equation 14 and considering the resulting action on $\boldsymbol{C} \times \{0\}$. Note that both vectors are killed by the form $dz - dy$. So, the claim is true at $(1,0)$. But both the contact field and the kernel of the 1-form are invariant under rotations, dilations, and vertical translations. Hence, by symmetry, the claim is true everywhere. ♠

Let Π be any contact plane. The restriction $\pi : \Pi \to \boldsymbol{C}$ naturally gives Π a complex structure – i.e. a canonical identification with $\boldsymbol{C}$. We can say that a Π-circle is an ellipse $E \subset \Pi$ such that $\pi(E)$ is a circle. We will see this structure on Π come up below when we discuss $\boldsymbol{C}$-circles.

3.2 $\boldsymbol{C}$-circles

A $\boldsymbol{C}$-circle is the image in $\mathcal{H}$ if the boundary of a $\boldsymbol{C}$-slice of $\boldsymbol{CH}^2$.

Lemma 3.3 *A straight line in $\mathcal{H}$ lies in a $\boldsymbol{C}$-circle iff the line is vertical.*

Proof: The set
$$\boldsymbol{CH}^2 \cap \{[z_1 : 0 : z_3]\} \tag{21}$$
is a $\boldsymbol{C}$-slice. The corresponding set in $\mathcal{H}$ is the vertical line $0 \times \boldsymbol{R}$. It now follows from translation symmetry that any vertical line in $\mathcal{H}$ lies in a $\boldsymbol{C}$-circle.

Conversely, suppose that L is a straight line lying in a $\boldsymbol{C}$-circle. Applying Heisenberg symmetries, we can assume that L goes through the origin. But then L corresponds to a $\boldsymbol{C}$-slice in $\boldsymbol{CH}^2$ containing $(0,0)$ and ∞. Such a $\boldsymbol{C}$-slice is unique and must be the one we have already considered, namely $0 \times \boldsymbol{R}$. ♠

Call a curve in $\mathcal{H}$ *non-straight* if it is not a straight line.

Lemma 3.4 *A loop X in $\mathcal{H}$ is a non-straight $\boldsymbol{C}$-circle if and only if X is a Π-circle contained in the contact plane Π through its own center of mass.*

Proof: For the purposes of this lemma, call a loop *good* if it is a Π-circle contained in the contact plane Π through its own center of mass. The one point compactification of the set

$$C\boldsymbol{H}^2 \cap \{(-1/2, w, 1)|\ |w| = 1\} \tag{22}$$

is a $\boldsymbol{C}$-slice. The corresponding $\boldsymbol{C}$-circle in $\mathcal{H}$ is $S^1 \times \{0\}$, where S^1 is the unit circle in $\boldsymbol{C}$. Note that the center of mass of X is the origin, and X is contained in $\boldsymbol{C} \times \{0\}$ and $\pi(X) = S^1$. So, $S^1 \times \{0\}$ is good and also a $\boldsymbol{C}$-circle.

Using Heisenberg symmetries, we see that every good loop is a $\boldsymbol{C}$-circle. Conversely, we can move a $\boldsymbol{C}$-circle by Heisenberg symmetries so that it contains the two points $(\pm 1, 0)$. But then, by uniqueness of $\boldsymbol{C}$-slices, the loop we get must be the one we first considered. Moreover, the Heisenberg symmetries preserve goodness. Hence, all $\boldsymbol{C}$-circles are good. ♠

3.3 $\boldsymbol{R}$-circles

A $\boldsymbol{R}$-circle is the image in $\mathcal{H}$ if the boundary of a $\boldsymbol{R}$-slice of $\boldsymbol{CH}^2$.

We call a curve $\gamma \subset \mathcal{H}$ *integral* if it is always tangent to the contact plane field. The $\boldsymbol{C}$-circles are not contact.

Lemma 3.5 *A straight line in $\mathcal{H}$ lies in an $\boldsymbol{R}$-circle if and only if it is contact.*

Proof: The set $\boldsymbol{CH}^2 \cap \boldsymbol{R}^2$ is the image of the parabola $x < -y^2/2$. The ideal boundary is the parabola $x = -y^2/2$. The corresponding $\boldsymbol{R}$-slice in $\mathcal{H}$ is the line $\boldsymbol{R} \times \{0\}$. This line is integral.

The image of $\boldsymbol{R} \times \{0\}$ under any Heisenberg symmetry is again an $\boldsymbol{R}$-circle. It is not hard to show that any straight line integral to the plane field can be obtained this way. Hence, every integral straight line lies in an $\boldsymbol{R}$-circle.

Conversely, any straight $\boldsymbol{R}$-circle corresponds to an $\boldsymbol{R}$-slice which can be moved to $\boldsymbol{CH}^2 \cap \boldsymbol{R}^2$ by symmetries which fix ∞. Hence every straight $\boldsymbol{R}$-circle is an integral straight line. ♠

We recall a definition from classical geometry. A *Lemniscate* is any curve which is similar to the one given, in polar coordinates, by $r^2 = \cos(2\theta)$. At the same time, a Lemniscate is what you get when you invert a square hyperbola $xy = a$.

Lemma 3.6 *A loop X in $\mathcal{H}$ is non-straight $\boldsymbol{R}$-circle if and only if X is integral and $\pi(X)$ is a lemniscate.*

Proof: Call X *good* if X is integral and projects to a lemniscate.

We apply the Heisenberg inversion I from Equation 17 to the $\boldsymbol{R}$-circle $\gamma = \boldsymbol{R} \times \{1\}$. Parametrizing this $\boldsymbol{R}$-circle by $(s, 1)$, we see that the first coordinate of $I(\gamma)$ is given by

$$\frac{-1}{s/2 + i/s}. \tag{23}$$

But the curve $s \to s/2 + i/s$ lies on the hyperbola $xy = 1/2$. Hence the first coordinate of $I(\gamma)$ is a Lemniscate. Hence $I(\gamma)$ is both good and an $\boldsymbol{R}$-circle.

The rest of the proof follows from the kind of symmetry used several times already. ♠

3.4 Bisectors and Spinal Spheres

Recall that a *bisector* in $\boldsymbol{CH}^2$ is the equidistant set between two points.

Lemma 3.7 *The set* $X = (\boldsymbol{R} \times \boldsymbol{C}) \cap \boldsymbol{CH}^2$ *is a bisector.*

Proof: Consider the points

$$p_\pm = (-1 \pm i : 0 : 1) \in \boldsymbol{CH}^2. \tag{24}$$

Given $q = (s : w : 1) \in X$, we compute

$$\langle p_\pm, p_\pm \rangle = -2; \qquad \langle q, p_\pm \rangle = (s-1) \pm i. \tag{25}$$

Therefore, the quantity

$$\cosh(d(p_\pm, q)) = \frac{\langle p_\pm, q\rangle \langle q, p_\pm \rangle}{\langle p_\pm, p_\pm \rangle \langle q, q \rangle} = \frac{(s-1)^2 + 1}{-2\langle q, q\rangle} \tag{26}$$

is the same for both points p_+ and p_-. ♠

A *spinal sphere* is the ideal boundary of a bisector. We call a spinal sphere in $\mathcal{H}$ *straight* if it contains ∞.

Lemma 3.8 *A surface in* $\mathcal{H}$ *is a straight spinal sphere if and only if it is a contact plane.*

Proof: The spinal sphere in $\mathcal{H}$ corresponding to the bisector we have just considered is $\boldsymbol{C} \times \{0\}$. This is certainly a contact plane. As above, the rest of the lemma follows from symmetry. ♠

The contact plane $\boldsymbol{C} \times \{0\}$ has a double (singular) foliation:

- The $\boldsymbol{C}$-circles $\{|z| = r\} \times \{0\}$ give one singular foliation.
- The straight $\boldsymbol{R}$-circles through the origin give the other one.

These give a natural polar coordinate system to the spinal sphere. By symmetry, these foliations exist on all the spinal spheres, not just the straight ones.

The non-straight spinal spheres are a bit messier to describe and I won't do it. They are all smooth spheres with these two singular foliations.

4 The Carnot Metric

The Heisenberg symmetries discussed above all cover similarities of $\boldsymbol{C}$. It turns out that there is a metric on $\mathcal{H}$ such that all Heisenberg symmetries are similarities. The metric is defined as follows. On each contact plane Π, we put the Euclidean metric which makes $\pi : \Pi \to \boldsymbol{C}$ an isometry. We then define the length of an integral curve to be the integral of the its speed! This definition makes no sense for non-integral curves. We define the *Carnot distance* between two points to be the infimal length of an integral curve connecting them.

Lemma 4.1 *The Carnot distance from* $(0,0)$ *to* $(0, 2\pi r^2)$ *is* $2\pi r$.

Proof: Use the contact form $dz - (xdy + ydx)$, and Green's Theorem, and the isoperimetric inequality. The best curve in this case is a helix which projects to the circle of radius r centered at the origin. ♠

Lemma 4.2 *The Carnot distance from* $(0,0)$ *to* $(r, 0)$ *is* r.

Proof: The segment $[0,r] \times \{0\}$ is the best integral connector in this case. ♠

Notice that when r is small, the distance from $(0,0)$ to $(0, \pi r^2)$ is huge in comparison to the Euclidean distance between the two points. Thus, the Carnot metric is not bi-lipschitz equivalent to the Euclidean metric (or any Riemannian metric) on $\mathcal{H}$. Indeed, it turns out that the Hausdorff dimension of $\mathcal{H}$, with respect to the Carnot metric, is 4-dimensional. This crazy-sounding statement is not so hard to prove. Using Lemmas 4.1 and 4.2 and symmetry, one can see that it takes about $O(\epsilon^{-4})$ Carnot-balls of diameter ϵ to fill up the unit Carnot ball.