Illustrated Proof of the Jordan Curve Theorem by **Rich Schwartz**
This note exposits J. W. Alexander's brilliant proof [**A**] of the Jordan Curve Theorem. Alexander's paper is the precursor to Alexander Duality, but you don't need to know about that stuff to understand the proof. Alexander's paper unfortunately does not have any pictures. This version has pictures, and simplifies and modernizes the proof. I learned everything about Alexander's paper from talking to Peter Doyle, and it was his idea to do this.

0. Preliminaries: A *Jordan curve* is the image J of the unit circle under a continuous injection into $\boldsymbol{R}^2$. Outline:

1. We discuss *cycles*.
2. We discuss *cycle approximation*.
3. We prove that $\boldsymbol{R}^2 - J$ has at least 2 components.
4. We prove the main technical result, Detour Lemma.
5. The Detour Lemma implies the Jordan Arc Theorem.
6. The D.L. and the J.A.T. imply $\boldsymbol{R}^2 - J$ has at most 2 components.

Conventions: *Component* means *path component*. S^o is the interior of S. A magenta set denotes the intersection of a red set and a blue set.

1. Cycles: A *cycle* is a planar graph with edges that are horizontal and vertical segments, having degree 2 or 4 at each vertex. A *cycle path* is a polygonal path with horizontal and vertical sides having no repeat sides. If the start and end vertex coincide, we call this a *cycle circuit*.

A cycle C separates $\boldsymbol{R}^2$ into two *sides*, making a checkerboard pattern. A cycle path c, in general position with respect to C, has endpoints on the same side of C if and only if $c \cap C$ is an even number of points. By Euler's Theorem, every cycle path in C extends to a cycle circuit in C.

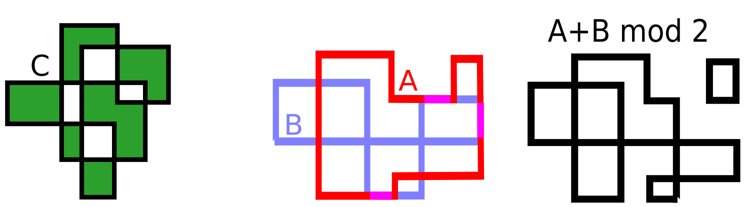

Figure 1: Cycle sides and cycle addition mod 2.

If A, B are cycles we interpret $A + B$ mod 2 as a cycle with underlying set $(A \cup B) - (B \cap A)$, adding vertices as needed. Likewise, we add vertices as needed to (suitable) unions of cycle paths so as to interpret them as cycles.

2. Cycle Approximation: A *coordinate rectangle* is a solid rectangle whose sides are horizontal and vertical. Each bounded side of a cycle, which we call a *bounded cycle side*, is a finite union of coordinate rectangles, and conversely. Therefore, if $S \subset \boldsymbol{R}^2$ is compact and $\epsilon > 0$ is given, we can find a bounded cycle side R such that $S \subset R^o$ and every point of R is within ϵ of a point of S. We call this *cycle approximation*.

3. At Least Two Components: We can find $a, b \in J$ and a square Γ such that a and b lie on opposite sides of Γ, and $J \cap \Gamma$ lies in the interior of the bottom side of Γ. Let J_R and J_B be the two arcs of J connecting a to b.

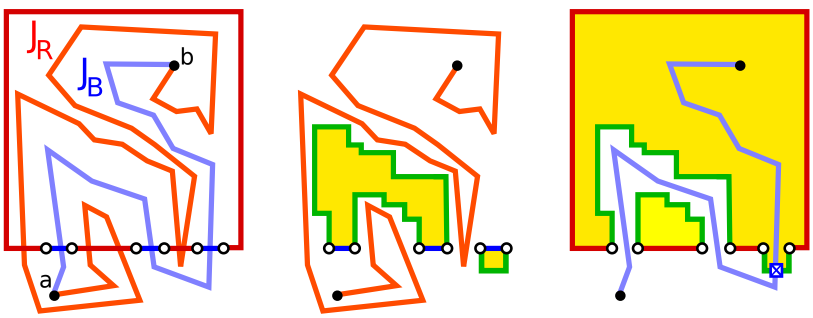

Figure 2: The sets involved in the proof

$J_R \cap \Gamma$ and $J_B \cap \Gamma$ are compact and disjoint, and hence are separated by a positive distance. By cycle approximation, we can place J_R (respectively J_B) inside the interior of a finite union of red (respectively blue) intervals so that red and blue are disjoint. Put another way we can partition the bottom edge of Γ into $2n+1$ intervals, alternating red and blue, so that J_R only hits red and J_B only hits blue. Color the rest of Γ the same color as the outermost intervals. Let Γ_R and Γ_B respectively be the red and blue parts of Γ.

If $\boldsymbol{R}^2 - J$ has just one component, we can connect the $2n$ points of $\Gamma_R \cap \Gamma_B$ in pairs by green cycle paths $A_1, ..., A_n$ that avoid $J \cup \{a, b\}$. (Our depiction of this impossible situation breaks in the right-hand figure.)

Let $A = A_1 \cup ... \cup A_n$. Consider the cycles $C_R = \Gamma_R \cup A$ and $C_B = \Gamma_B \cup A$. Since J_R avoids C_B the points a and b lie on the same side of C_B. Likewise a and b lie on the same side of C_R. Hence a transverse cycle path γ connecting a to b intersects C_R and C_B each an even number of times. But $\Gamma = C_R + C_B$ mod 2. Hence γ intersects Γ an even number of times. But a and b lie on opposite sides of Γ, a contradiction. Hence $\boldsymbol{R}^2 - J$ has at least 2 components.

4. The Detour Lemma: *Let $a, b \in \boldsymbol{R}^2 - (S \cup T)$ where $S, T \subset \boldsymbol{R}^2$ are compact. Suppose a, b are connected by cycle paths s, t with $s \cap T = t \cap S = \emptyset$ and $C = s \cup t$ a cycle. If $S \cap T$ lies entirely in one side of C, there is a cycle path in $\boldsymbol{R}^2 - (S \cup T)$ that connects a to b.*

Proof: The hypotheses of the result persist if we slightly enlarge S and T. So, by cycle approximation, it suffices to take S, T bounded cycle sides and to construct a connecting cycle path disjoint from $S \cup T^o$.

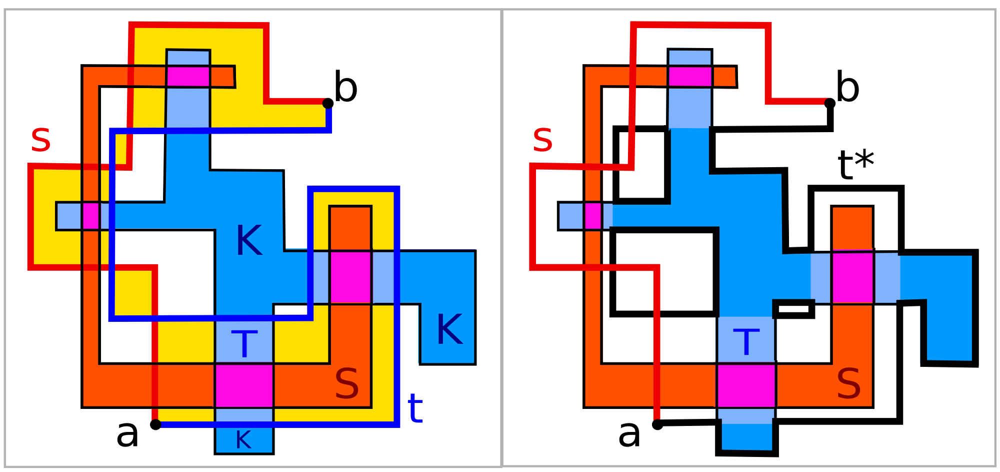

Figure 3: S (red), T (blue), C^* (thick red/black) and C^{**} (thick black).

Let K be the intersection of T with the side of C opposite $S \cap T$. Note that K is also a bounded cycle side, so that ∂K is a cycle. The cycle

$$C^* = (C + \partial K) \bmod 2 \tag{1}$$

is obtained from C by replacing all the edges of ∂K that lie in T with those that do not, so $C^* \cap T^o = \emptyset$. Let $C^{**} \subset C^*$ be the closure of $C^* - s$. Since $C^{**} \subset t \cup \partial K$, we have $C^{**} \cap (S \cup T^o) = \emptyset$. Extend s to a cycle circuit $s^* \subset C^*$. The cycle path $t^* = s^* - s \subset C^{**}$ connects a, b and avoids $S \cup T^o$. ♠

Remark: Our description of t^* can depend on some choices, but here is a canonical choice that has a simple description. Orient t from a to b. Follow t outside K, taking detours around the components of K, until you reach b. Basically, this works because after each detour you are further along t. A full proof with details might be longer than the proof involving Eulerian circuits.

5. The Jordan Arc Theorem: Let A be an arc of J. We prove that any $a, b \in \boldsymbol{R}^2 - A$ are joined by a cycle path in $\boldsymbol{R}^2 - A$.

By compactness we can partition A into sub-arcs $A_0, ..., A_n$ such that each A_i is contained in a disk that is disjoint from $a \cup b$. We can join a to b by a cycle path in $\boldsymbol{R}^2 - A_i$ for each $i = 0, ..., n$. By induction on n, we can join a to b by cycle paths s and t which respectively avoid $T = A_0 \cup ... \cup A_{n-1}$ and $S = A_n$. If needed, we perturb so that $s \cup t$ is a cycle. Since $S \cap T$ is a single point, $S \cap T$ lies on one side of $s \cup t$. By the Detour Lemma, we can connect a to b in $\boldsymbol{R}^2 - (S \cup T) = \boldsymbol{R}^2 - A$ by a cycle path.

6. At Most Two Components: We prove that some 2 of the arbitrarily chosen $a_1, a_2, a_3 \in \boldsymbol{R}^2 - J$ can be joined by a cycle path in $\boldsymbol{R}^2 - J$.

We take indices mod 3. Partition J into 3 Jordan arcs J_1, J_2, J_3. By the Jordan Arc Theorem we can join a_{i-1} to a_{i+1} by a cycle path s_i which avoids $J_{i-1} \cup J_{i+1}$. If needed, we perturb so that $C = s_1 \cup s_2 \cup s_3$ is a cycle. Let $u_i = J_{i-1} \cap J_{i+1}$. Two of these points, say u_1 and u_2, lie on the same side of C. The following sets satisfy the hypotheses of the Detour Lemma.

$$a = a_1, \quad b = a_2, \quad S = J_3, \quad s = s_3, \quad T = J_1 \cup J_2, \quad t = s_1 \cup s_2.$$

The main point is that $S \cap T = \{u_1, u_2\}$ lies on one side of $C = s \cup t$. Hence, we can connect $a_1 = a$ to $a_2 = b$ in $\boldsymbol{R}^2 - (S \cup T) = \boldsymbol{R}^2 - J$.

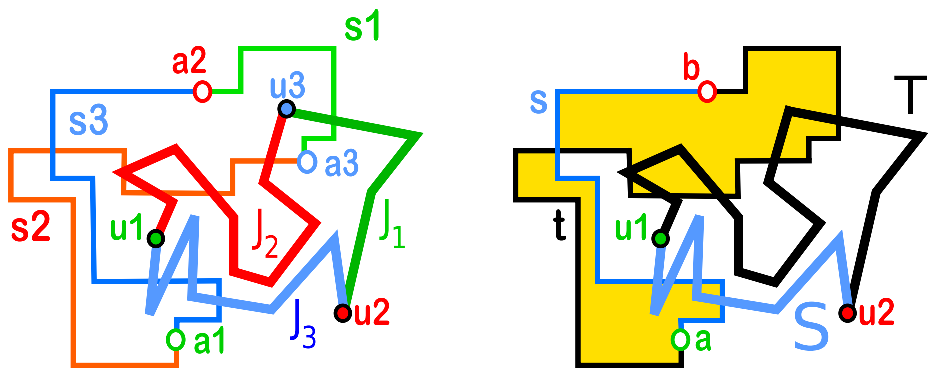

Figure 4: The sets involved in the proof

Reference: [**A**], J. A. Alexander, *A proof of Jordan's Theorem on Simple Curves*, Annals of Math, 1920.