Mobius Maps Preserve Circles

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September 10, 2013

The purpose of this note is to give a strange proof that Mobius transformations map circles to circles. Let $C_0 = \boldsymbol{R} \cup \infty$, considered as a subset of $\boldsymbol{C} \cup \infty$. The proof assumes that the Mobius transformations form a group G of homeomorphisms of $\boldsymbol{C} \cup \infty$. The proof is based on 4 additional properties.

1. For any circle C, there is some $T \in G$ such that $T(C_0) = C$.
2. If (a_1, a_2, a_3) and (b_1, b_2, b_3) are two triples of distinct points on C_0, then there is some $R \in G$ such that $R(C_0) = C_0$) and $R(a_i) = b_i$ for $i = 1, 2, 3$.
3. $R \in G$ is determined by where it takes 3 distinct points of C_0.
4. If γ is any non-circular loop, then there is some circle D such that $\gamma \cap D$ has cardinality at least 3.

Main Argument: Let $M \in G$ and let C be a circle. We could like to show that $\gamma = M(C)$ is a circle. Using Properties 1 and 4, it suffices to consider the case when $C = C_0$ and $\gamma = M(C_0)$ intersects C_0 in points b_1, b_2, b_3. Let $a_i = M^{-1}(b_i)$ for $i = 1, 2, 3$. Let $R \in G$ be given by Property 2. Then R and M agree on a_1, a_2, a_3. But then, by Property 3, $R = M$. But then $M(C_0) = R(C_0) = C_0$.

Property 1: Using similarities, we reduce to the case when C is the unit circle. The Mobius transformation $T(z) = (z + i)/(z - i)$ evidently maps C_0 into C, and the upper halfplane outside the unit disk, and the lower halfplane inside the unit disk. Since T is a homeomorphism, we must have $T(C_0) = C$.

Property 2: By the group property, it suffices to consider the case when $(b_1, b_2, b_3) = (0, 1, \infty)$. The map

$$T(z) = \frac{-(a_2 - a_3)(a_1 - z)}{(a_1 - a_2)(a_3 - z)}$$

is a Mobius transformation and has all the properties.

Property 3: Using Property 2, and the group property, it suffices to show that a Mobius transformation is the identity provided that it fixes $(0, 1, \infty)$. Starting with $T(z) = (az+b)/(cz+d)$, and plugging $T(0) = 0$ gives $b = 0$. Plugging in $T(1) = 1$ gives $a = c + d$. Plugging in $T(\infty) = \infty$ gives $c = 0$. We're left with $T(z) = z$.

Property 4: (The interesting one.) Suppose γ is noncircular. We can find a circle $C(0)$ that is contained in the region bounded by γ and intersects γ in at least two points, say a and b. Let $m(0)$ be the midpoint of one of the two arcs of $C(0)$ bounded by a and b. If $m(0) \in \gamma$ we are done. Otherwise, $m(0)$ lies in the interior of the region bounded by γ. Consider the family of circles $C(t)$ containing a and b. Choose the parameter t so that the distance from $m(t)$ to a tends to ∞ with t. But then, by continuity, there is some t such that $m(t) \in \gamma$. So, $C(t)$ is the desired circle.