Notes on Fourier Series and Modular Forms

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This is a meandering set of notes about modular forms, theta series, elliptic curves, and Fourier expansions. I learned most of the statements of the results from wikipedia, but many of the proofs I worked out myself. I learned the proofs connected to theta series from some online notes of Ben Brubaker. Nothing in these notes is actually original, except maybe the mistakes. My purpose was just to understand this stuff for myself and say it in terms that I like.

In the first chapter we prove some preliminary results, most having to do with single variable analysis. These assorted results will be used in the remaining chapters.

In the second chapter, we deal with Fourier series and lattices. One main goal is to establish that the theta series associated to an even unimodular lattice is a modular form. Our results about Fourier series are proved for functions that have somewhat artificial restrictions placed on them. We do this so as to avoid any nontrivial analysis. Fortunately, all the functions we encounter satisfy the restrictions.

In the third chapter, we discuss a bunch of beautiful material from the classical theory of modular forms: Eisenstein series, elliptic curves, Fourier expansions, the Discriminant form, the j function, Riemann-Roch.

In the fourth chapter, we will define the E_8 lattice and the Leech lattice, and compute their theta series.

1 Preliminaries

1.1 The Discrete Fourier Transform

Let

$$\omega = \exp(2\pi i/n) \tag{1}$$

be the usual nth root of unity.

Let $\delta_{ab} = 1$ if $a = b$ and otherwise 0. We have

$$\sum_{c=0}^{n-1} \omega^{c(b-a)} = n\delta_{ab}. \tag{2}$$

An equivalent version of Equation 2 is that the following two matrices

$$M = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & \omega^{-1} & \omega^{-2} & \cdots \\ 1 & \omega^{-2} & \omega^{-4} & \cdots \\ \cdots & & & \end{bmatrix} \tag{3}$$

$$M^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & \omega^{1} & \omega^{2} & \cdots \\ 1 & \omega^{2} & \omega^{4} & \cdots \\ \cdots & & & \end{bmatrix} \tag{4}$$

are inverses of each other.

The *discrete Fourier transform* is the linear transformation $\Psi : \boldsymbol{C}^n \to \boldsymbol{C}^n$ whose matrix is M. That is, given $z = (z_1, ..., z_n) \in \boldsymbol{C}^n$, we have $\Phi(z) = (Z_1, ..., Z_n)$, where

$$Z_k = \frac{1}{n}\sum_{j=0}^{n-1} z_j \omega^{-kj}. \tag{5}$$

Ψ^{-1} has the same form except that $(-kj)$ is replaced by (kj), and there is no $1/n$ out in front.

The map Ψ is a similarity relative to the Hermitian inner product

$$\langle (z_1, ..., z_n), (w_1, ..., w_n) \rangle = \sum_{i=0}^{n-1} z_i \overline{w_i} \tag{6}$$

That is

$$\langle Z, Z \rangle = \frac{1}{n}\langle z, z \rangle. \tag{7}$$

1.2 Fourier Series

Let $\mathcal{S}$ denote the space of complex-valued $\boldsymbol{Z}$-periodic functions on $\boldsymbol{R}$. That is, when $f \in \mathcal{S}$, we have $f(x) = f(x+1)$ for all $x \in \boldsymbol{R}$. Typically, the functions in $\mathcal{S}$ are taken to be Lebesgue integrable, but not necessarily continuous. For the applications, I'll always take continuous functions.

Define the Hermitian inner product

$$\langle f, g\rangle = \int_0^1 f\overline{g}\ dx. \tag{8}$$

This is defined for all integrable functions on $[0,1]$, though certainly is makes sense for functions in $\mathcal{S}$. We just restrict functions in $\mathcal{S}$ to this one interval. Conversely, if we have a function defined on $[0,1]$ which takes on the same values at the endpoints, then it is the restriction of a function in $\mathcal{S}$.

It is worth observing that the Hermitian inner product here is closely related to the one defined in the previous section. Given continuous functions $f, g : [0,1] \to \boldsymbol{C}$, we define

$$f_n = (f(0), f(1/n), f(2/n), ...); \qquad g_n = (g(0), g(1/n), g(2/n), ...).$$

These are vectors in $\boldsymbol{C}^n$. We have

$$\lim_{n\to\infty} \frac{1}{n}\langle f_n, f_n\rangle = \langle f, g\rangle. \tag{9}$$

The inner product on the left is the one defined in the previous section.

Let $f \in \mathcal{S}$. We define the *Fourier coefficients*

$$c_n(f) = \langle f, \exp(2\pi inx)\rangle. \tag{10}$$

Here $n \in \boldsymbol{Z}$. Assuming that the bi-infinite series $\{c_n\}$ converges absolutely, we define the *Fourier series*

$$\sum_{n\in\boldsymbol{Z}} c_n \exp(2\pi inx). \tag{11}$$

This series converges uniformly to a function. We will show that the new function actually equals f under certain conditions.

We call f nice if $|c_n| \leq C/n^2$ for some constant C, and

$$|f(x+h) - f(x)| \leq C|h|^e. \tag{12}$$

for some $e > 1/2$. This is meant to hold for all x, y, h. When $e = 1$, the function f is called Lipschitz.

Lemma 1.1 *If f is nice and $c_n(f) = 0$ for all n, then $f = 0$.*

Proof: If this lemma is false, we can normalize so that $\langle f, f\rangle = 2$. Since f is continuous, we have $\langle f_n, f_n\rangle > n$ for n large. But then $\langle F_n, F_n\rangle > 1$, where F_n is the DFT of f_n. Hence, some coefficient $C_k = C_k(f_n)$ of F_n is at least $1/\sqrt{n}$. That is:

$$|C_k(f_n)| = \left|\frac{1}{n}\sum_{j=0}^{n-1} f(j/n)\exp(-2\pi ik(j/n))\right| > \frac{1}{\sqrt{n}} \tag{13}$$

We first consider the case when $k \leq n/2$. This will allow us to use the inequality

$$\left|\exp(-2\pi ik/n) - 1\right| \geq \frac{k}{n}. \tag{14}$$

Let ϕ be the real-valued function which, for each $j = 0, ..., n-1$, identically equals $f(j/n)$ on $[j/n, (j+1)/n]$. Given that f is nice, we have $|f - \phi| < \epsilon_n/\sqrt{n}$ for some constant ϵ_n which tends to 0 as $n \to \infty$. Since $c_k(f) = 0$, we have $|c_k(\phi)| < \epsilon_n/\sqrt{n}$. That is,

$$\frac{\epsilon_n}{\sqrt{n}} \geq \left|\int_0^1 \phi(x)\exp(-2\pi ikx)dx\right| =$$

$$\left|\sum_{j=0}^{n-1} f(j/n)\int_{j/n}^{(j+1)/n} \exp(-2\pi ikx)\ dx\right| =$$

$$\left|\frac{-1}{2\pi ik}\sum_{j=0}^{n-1} f(j/n)\Big(\exp(-2\pi ik((j+1)/n) - \exp(-2\pi ik(j/n))\Big)\right| =$$

$$\left|\frac{-1}{2\pi ik} \times n \times C_k \times \Big(\exp(-2\pi ik/n - 1)\Big)\right| =$$

$$\frac{n}{2\pi k} \times |1 - \exp(-2\pi ik/n)| \times |C_k| \geq \frac{|C_k|}{2\pi}.$$

The last inequality is from Equation 14. Once $\epsilon_n < 1/(2\pi)$ we contradict Equation 13.

Ir remains to consider the case when $k > n/2$. Consider the function $g(x) = f(-x)$. Note that f has all Fourier coefficients 0 if and only if g does. We have $C_k(f_n) = C_{n-k}(g_n)$, so we can replace f by g and reduce to the case when $k \leq n/2$. ♠

Lemma 1.2 *if f is a nice function, then*

$$\sum_{n \in \boldsymbol{Z}} c_n \exp(2\pi i n x).$$

satisfies Equation 12.

Proof: Let g be the function under discussion. Define

$$g_n(x) = \sum_{|k| \leq n} c_n \exp(2\pi i n x).$$

Let $f_n = g - g_n$. The derivative of $c_k \exp(2\pi i k x)$ is at most $2\pi k |c_k|$. So, our conditions on $\{c_n\}$ guarantee that g_n has derivative at most

$$C_1 \sum_{k=1}^{n} \frac{1}{k} < C_2 \log(n).$$

Here C_1 and C_2 are two constants we don't care about. On the other hand,

$$|f_n(x + 1/n) - f_n(x)| \leq \sum_{|k| \geq n} |c_n| < C_3/n.$$

It suffices to take $h = 1/n$ in Equation 12. We have

$$|g(x + 1/n) - g(x)| \leq C_2 \log(n)/n + C_3/n < C_4(e)(1/n)^e$$

for any exponent $e < 1$. Here $C_4(e)$ is a constant that depends on e. ♠

Lemma 1.3 *If f is a nice function then*

$$f(x) = \sum_{n \in \boldsymbol{Z}} c_n \exp(2\pi i n x).$$

Proof: Let $g(x)$ be the function on the right hand side. By the previous result, g satisfies Equation 12. Let $\delta(m, n) = 1$ if $m = n$ and otherwise 0. We have the basic equation

$$\int_0^1 \exp(2\pi i m x) \exp(-2\pi i n x) \; dx = \delta(m, n)$$

From the uniform convergence of the sum defining $g(x)$, we can interchange the order of summation and integration to find that

$$\int_0^1 g(x)\exp(-2\pi inx)\ dx = \sum_{m\in\boldsymbol{Z}} c_m\delta(m,n) = c_n.$$

In short f and g have the same Fourier series. Let $h = f - g$. Then h satisfies Equation 12. Hence $h = 0$ by Lemma 1.1 ♠

Lemma 1.4 (Parseval) *Suppose f is a nice function. Then*

$$\langle f, f\rangle = \sum_{n\in\boldsymbol{Z}} |c_n|^2. \tag{15}$$

Proof: We have

$$\langle f, f\rangle =$$
$$\langle \sum c_m \exp(2\pi imx), \sum \overline{c}_n \exp(-2\pi inx)\rangle =$$
$$\sum_{m,n} \delta(m,n)c_m\overline{c}_n =$$
$$\sum_n |c_n|^2.$$

The uniform convergence allows us to interchange the integration and summation in our equations. ♠

1.3 The Zeta Function

Now we get to the application. The *zeta function* is the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{16}$$

It certainly converges for $s > 1$.

Consider the function

$$f(x) = 2\pi^2(x - 1/2)^2. \tag{17}$$

This is certainly a Lipshitz function on $[0,1]$ and hence satisfies Equation 12. Some routine calculations show

$$\langle f,f\rangle = \frac{\pi^4}{20}; \qquad c_0 = 2\pi^2/6; \qquad c_n = \frac{1}{n^2}.$$

These calculations show that f is a nice function. Feeding this information into Parseval's Identity gives

$$\zeta(4) = \frac{\pi^4}{90}. \tag{18}$$

Consider the function

$$f(x) = \frac{\pi^3 i(x-1/2)}{3}\Big(4(x-1/2)^2-1\Big) \tag{19}$$

The constants have been set up so that $f(0)=f(1)$. Also, f is clearly Lipschitz on $[0,1]$. Some routine calculations show that

$$\langle f,f\rangle = \frac{2\pi^6}{945}; \qquad c_0 = 0; \qquad c_n = \frac{1}{n^3}.$$

These calculations show that f is a nice function. Feeding this information into Parseval's Identity gives

$$\zeta(6) = \frac{\pi^6}{945}. \tag{20}$$

Similar techniques would establish formulae for $\zeta(k)$ for $k=8,10,12,\ldots.$ However, we only care about the case $k=4,6$.

1.4 Cauchy's Theorem

Let $f: U \to \boldsymbol{C}$ be a holomorphic function in a simply connected domain U. Let P be some polygon in U.

Theorem 1.5 (Cauchy) $\int_P f(z)dz = 0$.

Proof: Let D be the domain bounded by P, so that $P=\partial D$. Let $f=u+iv$. Letting dx and dy be the usual line elements, we can write

$$\int_{\partial D} f\ dz = \int_{\partial D}(u+iv)(dx+idy) = \int_{\partial D}(udx-vdy) + i\int_{\partial D}(vdx+udy).$$

By Green's theorem, the integral on the right-hand side equals

$$\int_D (u_y + v_x) dxdy + i \int_D (u_x - v_y) dxdy.$$

Both pieces vanish, due to the Cauchy–Riemann equations. ♠

Corollary 1.6 *For any $v \in \boldsymbol{R}$ we have*

$$\int_{\boldsymbol{R}} \exp(-\pi(x+iv)^2 \ dx = 1. \tag{21}$$

Proof: Let $f(z) = \exp(-\pi(z)^2)$. The function f is holomorphic in $\boldsymbol{C}$. Let $D = [-N, N] \times [0, v]$. Applying Cauchy's Theorem to ∂D, we have

$$\int_{-N}^{N} \exp(-\pi(x+iv)^2 \ dx = \int_{-N}^{N} \exp(-\pi x^2) \ dx + E_N. \tag{22}$$

Here E_N is the sum of the line integrals of f along the vertical sides of P. Given the way f decays as $\Re(z) \to \infty$, we have $\lim_{N\to\infty} E_N = 0$. Therefore

$$\int_{\boldsymbol{R}} \exp(-\pi(x+iv)^2 \ dx = \int_{\boldsymbol{R}} \exp(-\pi x^2) \ dx = I \tag{23}$$

Changing to polar coordinates,

$$I^2 = \int_{\boldsymbol{R}^2} \exp(-\pi x^2) dx = \int_0^\infty \int_0^{2\pi} r \exp(-\pi r^2) dr d\theta = \int_0^\infty \exp(-u) \ du = 1.$$

This completes the proof. ♠

1.5 The Cotangent Identity

When we consider Fourier expansions of modular forms, the following identity is very useful.

$$\frac{1}{z} + \sum_{d=1}^{\infty} \frac{1}{z-d} + \frac{1}{z+d} = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m; \qquad q(z) = \exp(2\pi i z). \tag{24}$$

The proof is contained in the two lemmas below.

Lemma 1.7

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{d=1}^{\infty} \frac{1}{z-d} + \frac{1}{z+d}$$

Proof: Let $L(z)$ and $R(z)$ respectively be the left and right hand sides of the above equation. Both these functions are meromorphic in $\boldsymbol{C}$ and invariant under the map $z \to z+1$. In a neighborhood of 0, we have $L(z) = 1/z + h_1(z)$ and $R(z) = 1/z + h_2(z)$, where both h_1 and h_2 are holomorphic functions which vanish at 0. But then $L-R$ is bounded in a neighborhood of 0. Hence $L-R$ is bounded in a neighborhood of each integer point. But both L and R are bounded away from the integer points. Hence $L-R$ is bounded. But then $L-R$ is constant. Given that h_1 and h_2 both vanish at 0, we have $L-R=0$. Hence $L=R$. ♠

Lemma 1.8

$$\pi\cot(\pi z) = \pi - 2\pi i \sum_{m=0}^{\infty} q^m(z).$$

Proof: Working within a small neighborhood, so that we can take square roots, we have

$$q^{\pm 1/2} = \exp(\pi i z) = \cos(\pi z) \pm i\sin(\pi z) \tag{25}$$

Hence

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{2i\cos(\pi z)}{2i\sin(\pi z)} = \frac{i(q^{1/2}+q^{-1/2})}{q^{1/2}-q^{-1/2}} =$$

$$\frac{i(q+1)}{q-1} = -i(q+1)(1+q+q^2...) = -i-2i(1+q+q^2...). \tag{26}$$

Multiplying through by π gives the result. ♠

1.6 The Discriminant of a Cubic

Consider the cubic

$$P(x) = P_0 + P_1x + P_2x^2 + P_3x^3. \tag{27}$$

$P(x)$ has multiple roots if and only if $P(x)$ and $Q(x) = P'(x)$ are not relatively prime. P and Q are not relatively prime if and only if there are polynomials a and b such that

$$aP + bQ = 0 \qquad \deg(a) < 2; \qquad \deg(b) < 3. \tag{28}$$

Writing

$$a(x) = a_0 + a_1 x; \qquad b(x) = b_0 + b_1 x + b_2 x^2, \tag{29}$$

we can express the condition as in Equation 28 as

$$M^t \cdot (a_0, a_1, b_0, b_1, b_2) = 0; \qquad M = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 & 0 \\ 0 & P_0 & P_1 & P_2 & P_3 \\ Q_0 & Q_1 & Q_2 & 0 & 0 \\ 0 & Q_0 & Q_1 & Q_2 & 0 \\ 0 & 0 & Q_0 & Q_1 & Q_2 \end{bmatrix} \tag{30}$$

Equation 30 has a nontrivial solution if and only if $\det(M) = 0$.

When $P = 4x^3 - g_2 x - g_3$, some algebra shows that

$$\det(M) = \Delta = g_2^3 - 27 g_3^2. \tag{31}$$

In short, P has multiple roots if and only if $\Delta = 0$. Δ is known as the *Discriminant* of P.

Now consider the two variable polynomial

$$F(x, y) = y^2 - P(x). \tag{32}$$

The zero set of F, denoted E_F and considered as a subset of the complex projective plane $\boldsymbol{CP}^2$ is called an *elliptic curve*. We have

$$\partial F/\partial y = 2y, \qquad \partial F/\partial x = P'(x). \tag{33}$$

Both partial derivatives vanish on the elliptic curve E_F if and only if $P(x)$ and $P'(x)$ have a common zero. Moreover, the point $[0 : 1 : 0] \in P^2(\boldsymbol{C})$ is always a regular point, as a direct calculation shows. So, the elliptic curve E_F is nonsingular if and only if $\Delta \neq 0$. As such, it is a smooth surface.

2 Higher Dimensional Fourier Series

2.1 Lattices

We equip $\boldsymbol{R}^n$ with its usual inner product. A lattice is a set of the form $\Lambda = T(\boldsymbol{Z}^n)$, where $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is an invertible linear transformation. The classical example, of course, is $\boldsymbol{Z}^n$ itself. The *dual lattice* Λ^* is defined to be the set of vectors v such that $v \cdot w \in \boldsymbol{Z}$ for all $w \in \Lambda$. For instance $(\boldsymbol{Z}^n)^* = \boldsymbol{Z}^n$. Let S^t denote the transpose of S.

Lemma 2.1 *If $\Lambda = T(\boldsymbol{Z}^n)$ then $\Lambda^* = (T^{-1})^t(\boldsymbol{Z}^n)$.*

Proof: Let $\Lambda' = (T^{-1})^t(\boldsymbol{Z}^n)$. We want to show that $\Lambda' = \Lambda^*$. Suppose first that $v \in \Lambda'$. We have

$$(T^{-1})^t(v) \cdot T(w) = v \cdot T^{-1}(T(w)) = v \cdot w. \tag{34}$$

Hence $v \in \Lambda^*$. Hence $\Lambda' \subset \Lambda^*$.

Suppose that $v \in \Lambda^*$. Then there is some vector v' such that $v = (T^{-1})^t(v')$. Equation 34 shows that $v' \cdot w \in \boldsymbol{Z}$ for all $w \in \boldsymbol{Z}^n$. Hence $v' \in \boldsymbol{Z}^n$. Hence $v \in \Lambda'$. Hence $\Lambda^* \subset \Lambda'$. ♠

Here are some definitions.

- Λ is *integral* if $\|v\|^2 \in \boldsymbol{Z}$ for all $v \in \Lambda$. (May not be standard term.)
- Λ is *even* if $\|v\|^2 \in 2\boldsymbol{Z}$ for all $v \in \Lambda$.
- Λ is *unimodular* if $\Lambda = T(\boldsymbol{Z})$ and T has unit determinant.
- Λ is *self-dual* if $\Lambda^* = \Lambda$.

Lemma 2.2 *If Λ is integral and unimodular, then Λ is self-dual.*

Proof: Since Λ is integral, we have $\Lambda \subset \Lambda^*$. Hence

$$\text{volume}(\boldsymbol{R}^n/\Lambda^*) \leq \text{volume}(\boldsymbol{R}^n/\Lambda),$$

with equality if and only if $\Lambda = \Lambda^*$. But T and $(T^{-1})^t$ both have determinant 1. Hence the two volumes are equal. Hence $\Lambda = \Lambda^*$. ♠

2.2 The Discrete Fourier Transform

Let Λ_m denote the lattice obtained by scaling Λ down by a factor of m. Likewise define Λ_m^*. Note that Λ_m/Λ and Λ_m^*/Λ^* are both finite abelian groups having m^n elements. Let $\boldsymbol{C}(\Lambda_m/\Lambda)$ be the vector space of maps from Λ_m/Λ to $\boldsymbol{C}$. Likewise define $\boldsymbol{C}(\Lambda_m^*/\Lambda^*)$.

The DFT is a linear map from $\boldsymbol{C}(\Lambda_m/\Lambda)$ to $\boldsymbol{C}(\Lambda_m^*/\Lambda^*)$. Given a map $f : \Lambda_m/\Lambda \to \boldsymbol{C}$ and $v \in \Lambda_m^*/\Lambda^*$ we define

$$\widehat{f}(v) = \frac{1}{m^n} \sum_{u \in \Lambda_m/\Lambda} f(u) \exp(-2\pi i u \cdot v). \tag{35}$$

This specializes to the DFT in the 1-dimensional case when $\Lambda = \boldsymbol{Z}$.

Lemma 2.3 *The DFT is a similarity relative to the standard Hermitian inner products on our vector spaces. The similarity constant is m^{-n}.*

Proof: We only care about the case $m > 1$. An orthonormal basis for $\boldsymbol{C}(\Lambda_m/\Lambda)$ is given by $\{f_v\}$. Here $f_v(v) = 1$ and otherwise $f_v(w) = 0$. We have

$$\widehat{f_v}(w) = \frac{1}{m^n} \exp(-2\pi i(v \cdot w)).$$

Hence

$$\langle f_u, f_v \rangle = \frac{1}{m^{2n}} \sum_{w \in \Lambda_n^*/\Lambda^*} \exp(2\pi i(w \cdot (u - v))$$

When $u = v$, this sum is clearly m^{-n}. When $u \neq v$, write $h = u - v$ and let

$$S = \sum \exp(2\pi i(w \cdot h)).$$

The sum is over $w \in \Lambda_m^*/\Lambda^*$. We want to prove that $S = 0$. Choose some $w_0 \in \Lambda_m^*/\Lambda^*$ so that $w_0 \cdot h \notin \boldsymbol{Z}$. We have

$$\exp(2\pi i w_0 \cdot h)S = \sum \exp(2\pi i((w + w_0) \cdot h)) =^* \sum \exp(2\pi i(w \cdot h)) = S.$$

The starred equality comes from the fact that Λ_m^*/Λ^* is a group. This equation clearly forces $S = 0$.

Now we know that $\{\widehat{f_v}\}$ is an orthogonal basis having norm m^{-n}. ♠

2.3 Fourier Series

Let Λ and Λ^* be as above. Let $|\Lambda|$ denote the volume of the torus $\boldsymbol{R}^n/\Lambda$. Let $\mathcal{S}$ denote the space of complex-valued (measurable) Λ-periodic functions. Let $f \in \mathcal{S}$. For any vector $v \in \Lambda^*$, we define

$$c_v = \frac{1}{|\Lambda|}\int_{\boldsymbol{R}^n/\Lambda} f(x)\exp(-2\pi i x\cdot v)\ dx. \tag{36}$$

We use Λ^* because we need a well defined function on $\boldsymbol{R}^n/\Lambda$.

Say that $f : \boldsymbol{R}^n/\Lambda \to \boldsymbol{C}$ is *nice* if f safisfies Equation 12 and $\{c_v\}$ decays faster than $\|v\|^{-N}$ for any N. (We are more restrictive here because we have a different application in mind.) The same argument as in the 1-dimensional case shows that $f = 0$ provided that f satisfies Equation 12 and $c_v = 0$ for all $v \in \Lambda^*$.

Lemma 2.4 *Suppose that $f : \boldsymbol{R}^n \to \boldsymbol{C}$ is a smooth and Λ-invariant function. Then f is nice.*

Proof: Let I be some multi-index. We define D_I to be the partial derative defined with respect to I. Repeated integration by parts gives the formula

$$c_v(D_I f) = v^I c_v. \tag{37}$$

Here we have set $c_v = c_v(f)$. Since $D_I(f)$ is continuous, there is a uniform bound on C_I on $|V^I c_v|$. Hence

$$|c_v| \leq \frac{C_I}{|v^I|}. \tag{38}$$

Here C_I is some constant that depends on I. Since we have a bound like this for every multi-index, we see that $|c_v|$ decays faster $\|v\|^{-N}$ for any N. ♠

Lemma 2.5 *If $f : \boldsymbol{R}^n \to \boldsymbol{C}$ is smooth and Λ-periodic, then*

$$f(x) = \sum_{v\in\Lambda^*} c_v \exp(2\pi i v\cdot x).$$

Proof: Let g be the function on the right hand side of the equation. We want to show that $f = g$. If f is smooth, then f certainly satisfies Equation 12. It now follows from the previous result that f is nice. The niceness condition implies that in fact g is smooth. In particular, g satiafies Equation 12. The rest of the proof is as in the 1-dimensional case. ♠

2.4 The Fourier Transform

The *Fourier transform* $\widehat{f}$ is defined as follows. For any $v \in \boldsymbol{R}^n$, we have

$$\widehat{f}(v) = \int_{\boldsymbol{R}^n} f(x) \exp(-2\pi i(x \cdot v))dx. \tag{39}$$

Here is a special example of interest to us. Define $f : \boldsymbol{R}^n \to \boldsymbol{R}$ by the formula

$$f(x) = \exp(-\pi|x|^2). \tag{40}$$

Here f is the classical Gaussian function.

Lemma 2.6 $\widehat{f} = f$.

Proof: We have

$$\widehat{f}(v) = \int_{\boldsymbol{R}^n} \exp(-|x|^2) \exp(2\pi i x \cdot v)\ dx.$$

Writing $x = (x_1, ..., x_n)$ and $v = (v_1, ..., v_n)$, we have $\widehat{f}(v) = I_1...I_n$, where

$$I_k = \int_{\boldsymbol{R}} \exp(-\pi x_k^2) \exp(2\pi i x_k v_k)\ dx_k.$$

Hence, it suffices to prove that $I_k = \exp(-v_k^2)$. Supressing the subscript k, we have

$$I = \int_{\boldsymbol{R}} \exp(-\pi x^2) \exp(2\pi i x v)\ dx =$$

$$\int_{\boldsymbol{R}} \exp(-\pi(x^2 + 2ixv)\ dx =$$

$$\int_{\boldsymbol{R}} \exp(-\pi(x + iv)^2 + v^2)\ dx =$$

$$\exp(-\pi v^2) \int_{\boldsymbol{R}} \exp(-\pi(x + iv)^2\ dx =$$

$$\exp(-\pi v^2),$$

as desired. In the last line, we applied Equation 21. ♠

2.5 Poisson Summation Formula

Suppose that $f : \boldsymbol{R}^n \to \boldsymbol{C}$ is a smooth and fast-decaying function. We assume that $|f^m(x)|$ decays faster that $\|x\|^{-N}$ for all m and all N. In short, all derivatives of f decay faster than polynomials.

Lemma 2.7 (Poisson Summation)

$$\sum_{v\in\Lambda} f(v) = \frac{1}{|\Lambda|} \sum_{v_*\in\Lambda^*} \widehat{f}(v_*).$$

Proof: Introduce the new function

$$g(x) = \sum_{v\in\Lambda} f(x+v). \tag{41}$$

The fast-decay condition on f implies that g exists, is smooth, and is Λ-periodic. Let $\{c_n\}$ be the Fourier coefficients of g. By Lemma 2.5, we have

$$g(x) = \sum_{v\in\Lambda^*} c_v \exp(2\pi i x \cdot v).$$

In particular

$$\sum_{v\in\Lambda} f(v) = g(0) = \sum_{v\in\Lambda^*} c_v. \tag{42}$$

We have

$$c_v = \frac{1}{|\Lambda|} \int_{\boldsymbol{R}^n/\Lambda} g(x) \exp(-2\pi i x \cdot v) dx =$$

$$\frac{1}{|\Lambda|} \sum_{v\in\Lambda} \int_{\boldsymbol{R}^n/\Lambda} f(x) \exp(-2\pi i x \cdot v) dx =$$

$$\frac{1}{|\Lambda|} \int_{\boldsymbol{R}^n} f(x) \exp(-2\pi i x \cdot v) dx = \frac{1}{|\Lambda|} \widehat{f}(v).$$

In short

$$c_v = \frac{1}{|\Lambda|} \widehat{f}(v). \tag{43}$$

Equations 42 and 43 together imply the lemma. ♠

2.6 Theta Series

Associated to each even lattice $\Lambda \subset \boldsymbol{R}^n$ we have the Θ-series:

$$\Theta(z) = \sum_{v \in \Lambda} \exp(\pi i z \|v\|^2) = \sum_{m=0}^{\infty} c_m q^m; \qquad q(z) = \exp(2\pi i z). \tag{44}$$

Here c_m is the number of vectors in Λ having square norm $2m$.

Since Λ is an even lattice, we have

$$\Theta(z+1) = \Theta(z). \tag{45}$$

We introduce the new function

$$F(s) = \Theta(is) = \sum_{v \in \Lambda} \exp(-\pi s \|v\|^2). \tag{46}$$

Lemma 2.8 *Suppose that Λ is even and unimodular. Then, for all $s > 0$, we have*

$$F(s) = s^{-n/2} F(1/s). \tag{47}$$

Proof: Consider the new lattice $s^{1/2}\Lambda$, obtained by scaling each vector of Λ by $s^{1/2}$. We have

$$(s^{1/2}\Lambda)^* = (s^{-1/2})\Lambda^* = (s^{-1/2})\Lambda. \tag{48}$$

The last equality comes from the fact that Λ is even unimodular, and hence self-dual.

Let's apply the Poisson Summation Formula to the function

$$f(x) = \exp(-\pi \|x\|^2)$$

and the lattice $s^{1/2}\Lambda$. We have

$$\sum_{v \in s^{1/2}\Lambda} f(v) = \sum_{v \in \Lambda} \exp(-\pi s \|v\|^2) = F(s). \tag{49}$$

Since $|\Lambda| = 1$ we have $|s^{1/2}\Lambda| = s^{n/2}$. We also have $\widehat{f} = f$. The Poisson Summation formula gives

$$F(s) = s^{-n/2} \sum_{v \in s^{-1/2}\Lambda} f(v) = s^{-n/2} \sum_{v \in \Lambda} \exp(-s^{-1} \|v\|^2) = s^{-n/2} F(1/s). \tag{50}$$

This establishes the lemma. ♠

Lemma 2.9 *Suppose that* Λ *is even and unimodular and* n *is even. Then*

$$\Theta(-1/z) = \mu z^{n/2}\Theta(z); \qquad \mu = (-i)^{-n/2}. \tag{51}$$

In particular,

$$\Theta(-1/z) = z^{n/2}\Theta(z) \tag{52}$$

when $n \equiv 0$ *mod* 8.

Proof: We observe that $F(1/s) = \Theta(-1/is)$. Now, we have

$$\Theta(is) = F(s) = s^{-n/2}F(1/s) = (-i)^{-n/2} \times (is)^{-n/2}\Theta(-1/is). \tag{53}$$

Our calculations shows that Equation 51 holds on the ray is, where $s > 0$. But both sides of Equation 51 are holomorphic functions. If they agree on a ray, they agree everywhere. ♠

3 Modular Forms

3.1 Basic Definitions

Let $\boldsymbol{H}^2 \subset \boldsymbol{C}$ denote the upper half plane. As usual, the modular group $SL_2(\boldsymbol{Z})$ acts on $\boldsymbol{H}^2$ by linear fractional transformations. A *modular form* is a holomorphic function $f : \boldsymbol{H}^2 \to \boldsymbol{C}$ such that

$$\sup_{\Im(z)>1} |f(z)| < \infty, \tag{54}$$

and

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z), \tag{55}$$

Here $2k$ is some non-negative integer. Defining $\tau(z) = (az+b)/(cz+d)$, we have an equivalent formulation of Equation 55:

$$f \circ \tau = \frac{1}{(\tau')^k} f \tag{56}$$

$2k$ is the *weight* of the modular form. Equation 55 is meant to hold for all elements of $SL_2(\boldsymbol{Z})$.

To verify Equation 55, it suffices to check the two conditions

$$f(z+1) = f(z); \qquad f(-1/z) = z^{2k} f(z). \tag{57}$$

One nice example of modular forms comes from certain even unimodular lattices. Given an even lattice Λ, recall that the theta series is defined as

$$\Theta_\Lambda(z) = \sum_{v \in \Lambda} \exp(\pi i \|v\|^2) = \sum_{m=0}^{\infty} c_m q^m. \tag{58}$$

Theorem 3.1 *Suppose that $\Lambda \subset \boldsymbol{R}^n$ is an even unimodular lattice and $n \equiv 0$ mod 8. Then Θ_Λ is a modular form of weight $n/2$.*

Proof: This is an immediate consequence of Equations 45 and 52. ♠

The 2 most famous examples of even unimodular lattices are the E_8 lattice in $\boldsymbol{R}^8$ and the Leech lattice in $\boldsymbol{R}^{24}$. We will construct these two lattices and identify the corresponding theta series.

3.2 Eisenstein Series

An equivalent way to think about modular forms is that they are holomorphic functions on the space $\mathcal{L}$ of lattices in $\boldsymbol{C}$ which obey the scaling law

$$g(a\Lambda) = a^{-2k} g(\Lambda). \tag{59}$$

One converts between this definition and the first definition using the rule $f(z) = g(\Lambda(1, z))$. Here $\Lambda(1, z)$ is the lattice generated by 1 and z.

The most classical examples are the Eisenstein series:

$$G_{2k}(\Lambda) = \sum_{\lambda \neq 0} \frac{1}{\lambda^{2k}}. \tag{60}$$

The sum takes place over nonzero lattice points. The function G_k is defined for $k = 2, 4, 6...$ and clearly satisfies Equation 59. Hence G_k is a modular form of weight $2k$.

3.3 Fourier Expansions

Let $f : \boldsymbol{H}^2$ be a modular form. Let $H \subset \boldsymbol{H}^2$ denote the set of points z with $\Im(z) > 1$. We write $z_1 \sim z_2$ if $z_1 - z_2 \in \boldsymbol{Z}$. The map

$$q(z) = \exp(2\pi i z). \tag{61}$$

gives a biholomorphic map from $H/\sim$ to the open unit disk Δ. By definition, $f \circ q^{-1}$ is holomorphic on $\Delta - \{0\}$ and bounded. Hence $f \circ q^{-1}$ is holomorphic on the unit disk. We therefore can write $f \circ q^{-1}$ as a Taylor series

$$f \circ q^{-1}(z) = \sum c_n z^n.$$

But then

$$f(z) = \sum c_n q^n(z). \tag{62}$$

This is the *Fourier series expansion* for f.

Lemma 3.2

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{63}$$

Here $\sigma_a(n) = d_1^a + ... + d_k^a$, where $d_1, ..., d_k$ are the divisors of n.

Proof: Differentiate Equation 24 $k-1$ times with respect to z:

$$(-1)^{k-1}(k-1)!\sum_{d\in\boldsymbol{Z}}\frac{1}{(z+d)^k}=-(2\pi i)^k\sum_{m=1}^{\infty}m^{k-1}q^m.$$

Multiply through, and use the fact that k is even, to get

$$\sum_{d\in\boldsymbol{Z}}\frac{1}{(z+d)^k}=\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}q^m. \tag{64}$$

Now we turn to Equation 63. The lattice $\Lambda(1,z)$ consists of all expressions $cz+d$ with $c,d\in\boldsymbol{Z}$. Therefore

$$G_k(z)=\sum_{c,d}\frac{1}{(cz+d)^k}=\sum_{d\neq 0}\frac{1}{d^k}+\sum_{c\in\boldsymbol{Z}-\{0\}}\sum_{d\in\boldsymbol{Z}}\frac{1}{(cz+d)^k}.$$

In the first sum, we mean to exclude $(c,d)=(0,0)$. When k is even, the sums simplify:

$$G_k(z)=2\zeta(k)+2\sum_{c=1}^{\infty}\sum_{d\in\boldsymbol{Z}}\frac{1}{(cz+d)^k}. \tag{65}$$

Equation 64 now gives

$$G_k(z)=2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}m^{k-1}q^{cm}. \tag{66}$$

But,

$$\sum_{c=1}^{\infty}m^{k-1}q^{cm}=\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n. \tag{67}$$

Equation 63 follows immediately from Equations 66 and 67. ♠

The two cases of Equation 63 of interest to us are $k=4$ and $k=6$. Knowing the values for $\zeta(4)$ and $\zeta(6)$, we can now say that

$$G_4(z)=\frac{\pi^4}{45}(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n). \tag{68}$$

$$G_6(z)=\frac{2\pi^6}{945}(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n). \tag{69}$$

3.4 The Weierstrass Function and Elliptic Curves

The Eisenstein series are closely connected with the Weierstrass $\wp$ function. Given a lattice Λ, we define

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{z^2} \right). \tag{70}$$

This is a meromorphic Λ-periodic function on $\boldsymbol{C}$, having a double pole at 0.

Letting $Q(z) = \wp(z) - 1/z^2$, we see that Q is holomorphic in a neighborhood of 0. Using the fact that Q is even, we see that $Q^{(k)}(0) = 0$ for k odd. Differentiating term by term, we see that $Q^{(2)}(0) = 3!G_4$ and $Q^{(4)}(0) = 5!G_6$, etc. Therefore, the Laurent series expansion for $\wp$ is

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 \dots. \tag{71}$$

Differentiating, we have

$$\wp'(z) = \frac{-2}{z^3} + 2 \cdot 3G_4 z + 4 \cdot 5G_6 z^3 + 6 \cdot 7G_8 z^5 \dots. \tag{72}$$

From these two expansions, we see that in a neighborhood of 0,

$$\wp^3(z) = \frac{1}{z^6} + 9G_4 \frac{1}{z^2} + 15G_6 + h_1(z) \tag{73}$$

$$(\wp'(z))^2 = \frac{4}{z^6} - 24G_4 \frac{1}{z^2} - 80G_6 + h_2(z). \tag{74}$$

$$(\wp'(z))^2 - 4\wp^3(z) = -60G_4 \frac{1}{z^2} - 140G_6 + h_3(z). \tag{75}$$

Here h_j, for $j = 1, 2, 3$, is a holomorphic function that vanishes at 0. But then

$$(\wp'(z))^2 - 4\wp^3(z) + 60G_4\wp(z) + 140G_6 \tag{76}$$

is a Λ-periodic function that vanishes at 0. Hence Equation 76 is identically zero. This establishes the classical differential equation for $\wp$, namely

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{77}$$

Here we have made the following definitions.

$$g_2 = 60G_4; \qquad g_3 = 140G_6. \tag{78}$$

The map $\Psi(z) = (\wp(x), \wp'(x))$ gives a map from $\boldsymbol{C}/\Lambda$ to the *Weierstrass elliptic curve* E_Λ having equation

$$y^2 = 4x^3 - g_2 x - g_3. \tag{79}$$

In order to make perfect sense of this map, we think of E as a projective curve in $P^2(\boldsymbol{C})$. For this we define $\Psi([0]) = [0:1:0]$, which the unique point where E intersects the line at infinity.

Lemma 3.3 *Ψ is a regular map.*

Proof: The function $\wp$ is an even function Λ-invariant function, and this gives us the equation

$$\wp(\lambda/2 + h) = \wp(\lambda/2 - h).$$

From this we see that $\wp'(\lambda/2) = 0$ as long as $\lambda/2$ is not actually a pole. There are 3 such points in $\boldsymbol{C}/\Lambda$. On the other hand $\wp'$ has an order 3 pole at $[0]$. Hence, $\wp'$ has exactly 3 zeros, counting multiplicity. Since we have already exhibited 3 distinct zeros, we see that $\wp'$ only vanishes at these three points, and $\wp''$ does not vanish at any of these points.

It remains to deal with the point $[0] \in \Lambda$. Now we have to remember that we are working in the projective plane. To compute the derivative we use the affine patch in which $[0:1:0]$ is the origin.

$$\Psi(z) = [\wp(z)/\wp'(z), 1, 1/\wp'(z)] = [-1/2z + h_1(z), 1, h_2(z)],$$

where h_1 and h_2 are holomorphic functions that vanish at 0. Hence the first coordinate of $\Psi'(0)$ is $-1/2$. Hence Ψ' is regular even at 0. ♠

We will show below that E_Λ is always a nonsingular elliptic curve. In this section, we will explicitly add this as a hypothesis.

Lemma 3.4 *Assume that E is a nonsingular curve. Then Ψ is surjective.*

Proof: Let $S = \Psi(\boldsymbol{C}/\Lambda) \subset E$. Since Ψ is a regular map, and E is nonsingular, S is open. Since Ψ is continuous, S is close. Since E is connected and S is nonempty, open, and closed, $S = E$. ♠

Lemma 3.5 *Assume that E is a nonsingular curve. Then Ψ is injective.*

Proof: Let $S \subset \boldsymbol{C}/\Lambda$ denote the set of points where Ψ is not injective. That is, if $s \in S$ there is some other $t \in S$ such that $\Psi(s) = \Psi(t)$. Note that $0 \notin S$. We will show that S is both open and closed. This shows that S must be empty.

To see that S is open, let $s \in S$. Let t be such that $\Psi(s) = \Psi(t)$. Since E is nonsingular, there are neighborhoods U_s and U_t such that $\Psi(U_s)$ and $\Psi(U_t)$ are respectively open neighborhoods of $\Psi(s)$ and $\Psi(t)$. Here U_s and U_t respectively are neighborhoods of s and t. But this shows that every point in a neighborhood of s belongs to S. Hence S is open.

To see that S is closed, suppose that $\{s_n\}$ is a sequence in S. Let $\{t_n\}$ be such that $\Psi(s_n) = \Psi(t_n)$ but $s_n \neq t_n$. let s and t be subsequential limits of $\{s_n\}$ and $\{t_n\}$ respectively. By continuity, $\Psi(s) = \Psi(t)$. Since Ψ is a regular map, and E is nonsingular, Ψ is locally injective. Hence s_n and t_n cannot get too close. Hence $s \neq t$. Hence $s \in S$. This shows that S is closed. ♠

Corollary 3.6 *Assume that E is a nonsingular curve. Then $\Psi : \boldsymbol{C}/\Lambda \to E$ is a biholomorphism.*

Proof: If E is a nonsingular curve, then E is a Riemann surface. In this case, $\Psi : \boldsymbol{C}/\Lambda \to E$ is a bijective map which is also biholomorphic. But note also that Ψ is regular. By the inverse function theorem, Ψ^{-1} is also holomorhic. In short, when E is nonsingular, $\Psi : \boldsymbol{C}/\Lambda \to E$ is a biholomorphism. ♠

The map Ψ is known as the *Weierstrass parametrization* of E. The Elliptic curve E turns out to be a group in a natural way, just as $\boldsymbol{C}/\Lambda$ is a group. The map Ψ is also a group isomorphism. This fact is also not hard to prove once one realizes that the group laws on both $\boldsymbol{C}/\Lambda$ and on E are biholomorphic.

3.5 The Discriminant Form

We have

$$g_2(z) = \frac{4\pi^4}{3}(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n), \quad g_3(z) = \frac{8\pi^6}{27}(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n). \quad (80)$$

Observe that

$$\Delta = g_2^3 - 27g_3^2 \quad (81)$$

has no constant term in its Fourier series expansion. Forms with this property are called *cusp forms*. Δ is called the *Discriminint form*.

Lemma 3.7 *Δ is nonzero on $\boldsymbol{H}^2$. Hence the Weierstrass elliptic curve E_Λ is always nonsingular.*

Proof: Let $S \subset \boldsymbol{H}^2$ denote those z for which $\Delta(z) = 0$. We want to show that S is empty. A calculation shows that Δ is sometimes nonzero, so S is a discrete set. Suppose that $z \in S$. We can find a sequence $z_n \to z$ with $\Delta(z_n) \neq 0$. Let Λ_n be the corresponding lattice and let E_n be the corresponding elliptic curve. Let $\Psi_n : \boldsymbol{C}/\Lambda_n \to E_n$ be the Weierstrass map.

Consider the function

$$\beta(z) = \frac{g_2^3(z)}{g_3^2(z)} = 27 + Cq + ... \quad (82)$$

Note that β is the quotient of two modular forms of the same weight. Hence β is actually an $SL_2(\boldsymbol{Z})$ invariant meromorohic function.

From the Fourier series expansions, we can see that $C \neq 0$. Hence β is regular in a neighborhood of ∞. By the Inverse Function Theorem, β takes on all values sufficiently close to 27 in every neighborhood of ∞. At the same time, $\beta(z) = 27$ if and only if $\Delta(z) = 0$. So, $\beta(z_n) \to 27$. Hence, there is some sequence $\{z_n'\}$ such that $\beta(z_n') = \beta(z_n)$ and $\Im(z_n') \to \infty$. Let Λ' be the lattice associated to z_n'. Let E_n' be the corresponding elliptic curve.

Since $\beta(z_n) = \beta(z_n')$, there is some change of coordinates of the form $(x, y) \to (ax, by)$ which carries E_n to E_n'. Hence E_n' and E_n are biholomorphic. But then $\boldsymbol{C}/\Lambda_n$ and $\boldsymbol{C}/\Lambda'$ are biholomorphic. However, once n is large enough, the lattices Λ_n and Λ_n' are not the same up to scale. But then $\boldsymbol{C}/\Lambda_n$ and $\boldsymbol{C}/\Lambda_n'$ cannot be biholomorphic. This is a contradiction. The only way out is that Δ never vanishes. ♠

3.6 The j Function

The j function is defined to be

$$j(z) = 1728\frac{g_2^3(z)}{\Delta(z)} = \frac{1728g_2^3}{g_2^3 - 27g_3^2} \tag{83}$$

The crazy normalizing constant is present so that j has residue 1 at ∞. More precisely, the Fourier series starts out

$$j(z) = \frac{1}{q} + 744 + 196884q... \tag{84}$$

One can see this just by manipulating our equations for g_2 and g_3 in a straightforward way. The number 196884 is closely related to the Leech lattice.

Let $X = \boldsymbol{H}^2/SL_2(\boldsymbol{Z}) \cup \infty$. The set X has a structure of a Riemann surface. The map $z \to q(z)$ gives a coordinate chart in a neighborhood of ∞.

Lemma 3.8 *j is a biholomorphism from X to $\boldsymbol{C} \cup \infty$.*

Proof: Being the ratio of two modular forms of the same weight j is an $SL_2(\boldsymbol{Z})$ invariant function. Hence $j : X \to \boldsymbol{C} \cup \infty$ is well defined and holomorphic. One checks from the Fourier expansion that Δ has a simple 0 at ∞ and g_2 does not vanish at ∞. Hence j has a simple pole at ∞. By the inverse function theorem, j is a homeomorphism in a neighborhood of ∞.

Being a holomorphic map, $j(X)$ is open. Since X is closed, $j(X)$ is closed. Hence $j(X)$ is both open, closed, and nonempty. Hence $j(X) = \boldsymbol{C} \cup \infty$.

To prove that j is injective, note that

$$1 - 1/j(z) = \beta(z) = \frac{g_2^3(z)}{g_3^2(z)}.$$

So, the value of j determines the value of β. But if $\beta(z_1) = \beta(z_2)$ then the corresponding elliptic curves are equivalent, as we discussed in the proof of Lemma 3.7. But no two lattices in $\boldsymbol{H}^2/SL_2(\boldsymbol{Z})$ are equivalent. Hence j is injective.

Now we know that j is a bijection. If j was not regular at some point z_0, then we would have an expansion

$$j(z) - j(z_0) = Cz^2 + ...$$

and j could not be injective in a neighborhood of z. Hence j is regular at each point. It now follows from everything said that $j : X \to \boldsymbol{C} \cup \infty$ is a biholomorphism. ♠

3.7 The Riemann-Roch Theorem

Let V_k denote the vector space of weight k modular forms. Let V_k^0 denote the vector space of weight k cusp forms. The Riemann-Roch Theorem is quite a general result, and here we are just stating a special case.

Theorem 3.9 (Riemann-Roch) *The following is true.*

- $\dim(V_2) = 0$.
- $\dim(V_k) = 1$ *for* $k = 0, 4, 6, 8, 10$.
- $\dim(V_{k+12}) = \dim(V_k) + 1$ *for all even* $k \geq 0$.
- $\dim(V_k) = \dim(V_k^0) + 1$ *for all even* $k \geq 4$.

We will prove this result through a series of smaller lemmas. The key to the whole proof is the discriminant form Δ.

Lemma 3.10 $\dim(V_{12}^0) = 1$. *That is, every weight* 12 *cusp form is proportional to the discriminant form.*

Proof: Let A be a weight 12 cusp form that is not proportional to Δ. Let $f = B/\Delta$. Since Δ does not vanish and has only a simple zero at ∞, the function f has no poles at all. But f is a holomorphic function on $X = \boldsymbol{H}^2/SL_2(\boldsymbol{Z}) \cup \infty$. The function $g = f \circ j^{-1}$ is therefore a holomorphic function on $\boldsymbol{C} \cup \infty$ without poles. But then g is constant. Hence f is constant. ♠

Lemma 3.11 $\dim(V_k^0) = 0$ *for* $k = 2, 4, 6, 8, 10$.

Proof: Let $A \in V_k^0$ be a supposed cusp form. Then the function

$$f = A^{12}/\Delta^k$$

is a holomorphic function of the same quotient X which vanishes at ∞. But then f is identically zero. Hence A is identically zero. ♠

Lemma 3.12 $\dim(V_k) = 1$ *for* $k = 4, 6, 8, 10$.

Proof: If $\dim(V_k) > 1$ then we can find two linearly independent forms A and B of weight k. But then some suitable linear combination $aA + bB$ has no constant term in its Fourier series expansion and yet is nontrivial. This contradicts the previous result. Hence $\dim(V_k) \leq 1$ for $k = 4, 6, 8, 10$. The existence of the forms G_4, G_6, G_4^2, and G_4G_6 show that $\dim(V_k) \geq 1$ for $k = 4, 6, 8, 10$. ♠

Lemma 3.13 $\dim(V_0) = 1$ *and* $\dim(V_2) = 0$.

Proof: The only members of V_0 are the constant functions, since there are no bounded holomorphic function on $X = \boldsymbol{H}^2/SL_2(\boldsymbol{Z}) \cup \infty$.

Suppose there is some nontrivial $A \in V_2$. We normalize so that $A = 1 + C_1q + ...$ Then the series expansion for $A^2 \in G_4$ starts $1 + 2C_1q + ...$ Since $\dim(V_4) = 1$, we see from Equation 68 that $C_1 = 120$. On the other hand $A^3 \in G_6$ and the series expansion starts out $1 + 3C_1q + ...$ This gives $C_1 = -168$, a contradiction. ♠

Lemma 3.14 $\dim(V_k) = \dim(V_k^0) + 1$ *for all even* $k \geq 4$.

Proof: Suppose $A, B \in V_k$ are two disctinct elements, Considering the Fourier series expansion, we can find a constant c to that $A + cB$ is a cusp form. This shows that $\dim(V_k/V_k^0) \leq 1$. Hence $\dim(V_k) \leq \dim(V_k^0) + 1$. From the analysis above we see that $V_k - V_k^0$ contains a nontrivial element for all even $k \geq 4$. ♠

Lemma 3.15 $\dim(V_{k+12}) = \dim(V_k) + 1$ *for all even* $k \geq 0$.

Proof: We have a map $T : V_k \to V_{k+12}^0$. The map is $T(A) = A\Delta$, where Δ is the discriminant form. This map is evidently a vector space isomorphism. The point is that the inverse map is given by $B \to B/\Delta$. Now we see that $\dim(V_{k+12})^0 = \dim(V_k)$. The previous result finishes the proof. ♠

4 Two Famous Lattices

4.1 The E_8 Lattice

The E_8-lattice Λ is an 8-dimensional even unimodular lattice. It turns out to be unique. A vector $(v_1, ..., v_8) \in \boldsymbol{R}^8$ belongs to Λ iff $v_1 + ... + v_8 \in 2\boldsymbol{Z}$ and all coordinates of $2v$ are integers of the same parity.

Lemma 4.1 *Λ is even.*

Proof: Let $v \in \Lambda$. Let $w = 2v$. It suffices to prove that $\|w\|^2 \in 8\boldsymbol{Z}$. We have $w_1 + ... + w_8 \in 4\boldsymbol{Z}$. Suppose first that all the coordinates of w are even. The condition on the sum forces an even number of the coordinates to be congruent to 2 mod 4. From here the result is obvious. Any odd number x has the property that $x^2 \equiv 1$ mod 8. So, in the odd case, the sum of the squares of 8 odd numbers must be congruent to 0 mod 8. ♠

Lemma 4.2 *Λ is unimodular*

Proof: Consider the linear transformation T given by

$$T^t = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \tag{85}$$

The rows of T^t all belong to Λ and so the columns of T all belong to Λ. Hence $T(\boldsymbol{Z}^8) \subset \Lambda$. A calculation shows that $\det(T) = 1$. To finish the proof, we want to see that $X = \Lambda - T(\boldsymbol{Z}^8)$ is empty. Suppose X is nonempty. Since $T(\boldsymbol{Z}^8)$ contains $(1/2, ..., 1/2)$, we can say that X contains an integer vector. By inspection, we see that $T(\boldsymbol{Z}^8)$ contains $2e_k$ for each standard basis vector e_k. Hence, if X is nonempty, X contains a vector $v = (v_1, ..., v_k)$ with $v_i \in \{0, 1\}$ for each i. But this forces $v = (0, ..., 0)$ or $v = (1, ..., 1)$. But both vectors clearly belong to $T(\boldsymbol{Z}^n)$. This is a contradiction. ♠

Since Λ is even, the distance between every pair of vectors in Λ is at least $\sqrt{2}$. So, if we place balls of radius $\sqrt{2}$ so that their centers are the points of Λ, we get a packing of balls. The following result shows that each ball is tangent to exactly 240 other balls.

Lemma 4.3 *There are* 240 *vectors of* Λ *having norm* $\sqrt{2}$*.*

Proof: There are only 2 kinds of vectors in Λ of norm $\sqrt{2}$:

- $(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$ and permutations.
- $(\pm 1/2, ..., \pm 1/2)$ and permutations.

In the second case, the number of $(-)$ signs must be even and likewise the number of $(+)$ signs must be even. An easy enumeration shows that there are 240 such vectors. ♠

Theorem 4.4 *Suppose that* Λ *is the* E_8 *lattice. Then*

$$G_4 = \frac{\pi^4}{45}\Theta_\Lambda.$$

Proof: By Lemma 4.3, we have

$$\Theta_\Lambda = 1 + 240q + ... \tag{86}$$

Equation 68 gives us

$$G_4(z) = \frac{\pi^4}{45}\Big(1 + 240q + ...\Big) \tag{87}$$

Let V_4 denote the vector space of weight 4 modular forms. We know that $\dim(V_4) = 1$. Hence the left and right hand sides of the equation in Theorem 4.4 are proportional. Comparing Equations 86 and 87, we see that the left and right hand sides agree at infinity. Hence, they agree everywhere. ♠

4.2 The Leech Lattice

The Leech lattice has many definitions. Here is one definition. Order the length 24 binary strings lexicographically. Pick a subset as follows. Start with 0...0. At each stage of the construction, choose the first string which differs by at least 8 bits from all strings on the list. This process generates 2^{12} binary strings. These strings make a 12 dimensional subspace $\mathcal{G}$ of $(\boldsymbol{Z}/2)^{24}$. Every two strings of $\mathcal{G}$ differ by at least 8 bits. $\mathcal{G}$ is known as the *Golay* $(24, 12)$ *code*.

Given $s \in \mathcal{G}$ and any integer m, let $L(s,m) \subset \boldsymbol{R}^{24}$ denote those vectors $v = (v_1, ..., v_{24})$ such that

- $v_1 + ... + v_{24} = 4m$.
- $v_k - m \equiv 2s_k \bmod 4$.

Then

$$\Lambda = \bigcup_{s \in \mathcal{G}} \bigcup_{m \in \boldsymbol{Z}} L(s,m). \tag{88}$$

Theorem 4.5

$$\Theta_{24} = \Theta_8^3 - 720\ (2\pi)^{-12}\Delta = 1 + 196560q^2 + ... \tag{89}$$

So, there are 196560 *vectors of length* 2 *in the Leech lattice.*

Proof: The Leech lattice $\Lambda \subset \boldsymbol{R}^{24}$ is even and unimodular, and the shortest vectors have length 2. Let Θ_{24} be the corresponding theta series. Since Λ is even and unimodular, Θ is a modular form of weight 12. That is, $\Theta_{24} \in V_{12}$. Since Λ has no vectors of square norm 2, we have

$$\Theta_{24} = 1 + (0 \times q) + m_2 q^2 + m_3 q^3 + ... \tag{90}$$

Let Θ_8 be the theta series for the E_8 lattice. Since Θ_8^3 is not a cusp form, the two forms Θ_8 and Δ span V_{12}. So, Δ_{24} is some linear combination of Θ_8^3 and Δ. We observe that

$$\Theta_8^3 = 1 + 720q...; \qquad (2\pi)^{-12}\Delta = q - 24q^2... \tag{91}$$

Evidently the only possibility is given by the theorem. ♠