Square Turning Maps and their Compactifications

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Abstract

In this paper we introduce some infinite rectangle exchange transformations which are based on the simultaneous turning of the squares within a sequence of square grids. We will show that such noncompact systems have higher dimensional dynamical compactifications. In good cases, these compactifications are polytope exchange transformations based on pairs of Euclidean lattices. In each dimension $8m+4$ there is a $4m+2$ dimensional family of them. Here $m = 0, 1, 2, ...$ We studied the case $m = 0$ in depth in [**S1**].

1 Introduction

1.1 Background

A *piecewise isometry* is a map defined on a typically polyhedral subset of Euclidean space. The domain is partitioned into smaller polyhedra in such a way that the map is defined, and an isometry, when restricted to the interior of each polyhedron in the partition. (One could make a similar definition for piecewise affine maps.) These maps often have a special beauty and combinatorial feel to them, but currently they are very far from being understood. In particular, there is a scarcity of examples above dimension 1, and especially above dimension 2.

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The simplest examples of piecewise isometries are 1-dimensional interval exchange transformations. See [**K**], [**R**], [**Y**], [**Z**] for an important but small sample of this large field. The paper [**H**] is an early paper on rectangle exchange transformations. The papers [**AE**], [**AG**], [**AKT**], [**Go**], [**Low1**], [**Low2**], [**LVK**], [**T**], all treat closely related sets of systems with k-fold symmery for k typically equal to 5, 7, or 8. Some definitive theoretical work concerning the (zero) entropy of such maps is done in [**GH1**], [**GH2**], and [**B**]. My monographs [**S1**] and [**S2**] have additional references.

The purpose of this paper is to study the dynamics of maps defined in terms of square grids. Given some $s > 0$ and some $z \in \boldsymbol{C}$, let $G_{s,z}$ be the (usual) infinite grid of squares in the plane such that z is a vertex of the grid and the sides of the squares are parallel to s and is. In particular, the squares have side length s. Let $R_{s,z}$ denote the map which rotates each square of $G_{s,z}$ clockwise by $\pi/2$ radians. The map $R_{s,z}$ is only defined on the interiors of the squares of $G_{s,z}$.

The map $R_{s,z}$ is not very interesting from a dynamical perspective, because $R^4_{s,z}$ is the identity map. However, the compositions of the form

$$R_{S,Z} := R_{s_1,z_1} \circ ... \circ R_{s_n,z_n} \quad (1)$$

can have quite intricate behavior. Here $S = \{s_1, ..., s_n\}$ and $Z = \{z_1, ..., z_n\}$ are the data the the composition. When n is divisible by 4, the map $R_{S,Z}$ is a piecewise translation. More specifically, $R_{S,Z}$ is an infinite rectangle exchange transformation in this case.

One nice feature of (planar) piecewise translations is that they define a natural family of convex polygons in the plane. When $R = R_{S,Z}$ is a piecewise translation, every periodic point p of R is contained in a maximal open convex polygon I_p, called a *periodic island*, such that R is defined entirely on I_p and periodic there. Since R is a rectangle exchange, the polygon I_p is in fact an open rectangle.

Probably the two most basic questions one can ask about the maps in Equation 1 are about their periodic points and about their unbounded orbits. In this paper we will talk mainly about periodic points, though a few of our results have to do with the existence of unbounded orbits. Our interest in understanding these basic questions for a special case is what led us to the idea of compactifying the systems.

1.2 A Motivating Example

The original motivation behind this paper was to understand the dynamics of the maps

$$A_s = R_{1,0} R_{s,0} R_{1,0} R_{s,0}. \tag{2}$$

In [**S1**], we called A_s the *alternating grid system* because we generate the dynamics by alternately turning the squares in the side-length-1 grid and the squares in the side-length-s grid. The origin is a vertex of both grids.

Let Q_s denote the square of $G_{s,0}$ whose bottom left vertex is the origin. Figure 1.1 shows how Q_s is tiled by the periodic islands for $s = 36/31$. The continued fraction expansion of s is $0:1:1:2:3:1:2$.

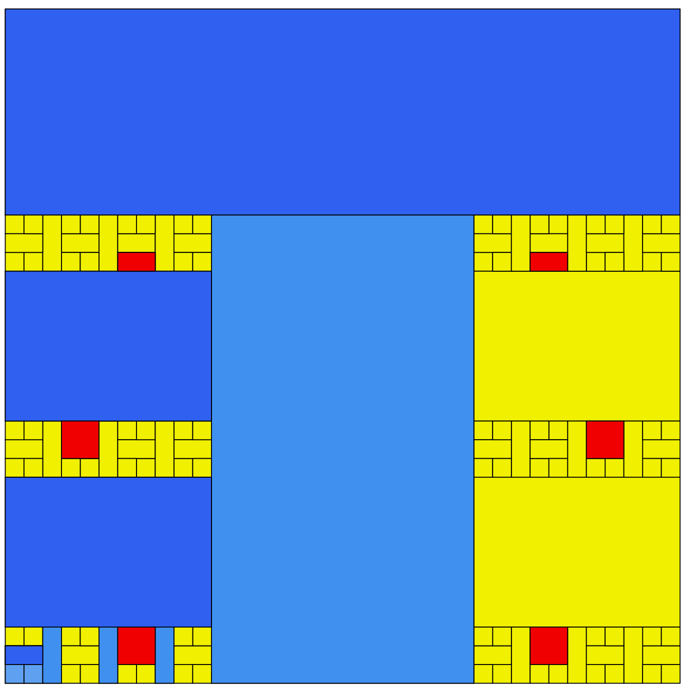

Figure 1.1: Periodic island tiling of Q_s for $s = 36/61$.

If we count the number of blue islands in the picture which alternately abut the two coordinate axes, we see that these numbers encode the continued fraction expansion of s. The red islands suggest "blemishes". It seems that these red islands should be subdivided into smaller tiles in order to make the overall pattern perfect.

Figure 1.2 shows the same kind of picture for the parameter $s = 55/89$. Here, the C.F.E. is $0 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 2$.

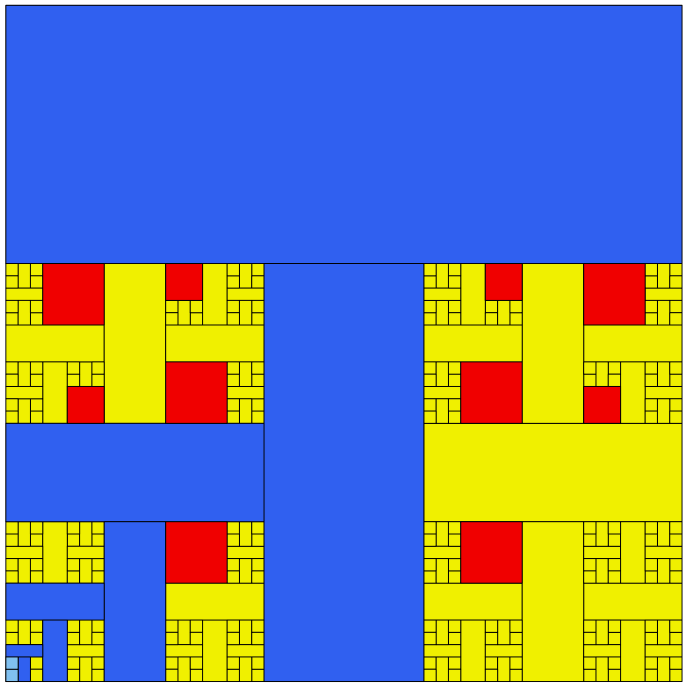

Figure 1.2: Periodic island tiling of Q_s for $s = 55/89$.

Pictures such as these suggest

Conjecture 1.1 *Let $s \in R_+$. Almost every point of A_s is periodic. Inside Q_s one can find a sequence of periodic islands, alternately sharing edges with the two coordinate axes, which encodes the continued fraction expansion of s.*

The difficulty in proving Conjecture 1.1 is that the periodic orbits corresponding to small islands seem to be quite long and also to have huge diameters. (The mysterious red islands are also an obstacle to our understanding.) This leads one to the idea of trying to compactify the system, in order to bring all the orbits into view, so to speak. We took this approach (with partial success) in [**S1**] for A_s and here we will do it more generally.

1.3 Dynamical Results

The alternating grid system is an example of a composition from Equation 1 which involves 2 grids. For the purpose of stating some results, we formalize the idea of such compositions. Also, for convenience, we take the parameter s to be irrational. In the rational case, all the orbits are periodic. There is a lot to say about the rational case, but we will not say it here.

Let G be the free product $(\boldsymbol{Z}/4) * (\boldsymbol{Z}/4)$. Let A_1 and A_2 be the usual generators of G. Let $G^0 \subset G$ denote the index 4 subgroup consisting of words whose total exponent is divisible by 4. The word $(A_1A_2)^2$, which corresponds to the alternating grid system, is an example of a word in G^0.

Given $S = \{s_1, s_2\}$ and $Z = \{z_1, z_2\}$, we have a representation of G into map($\boldsymbol{C}$) which sends A_j to R_{s_j,z_j}. By scaling and translating the plane, an operation which does not effect the dynamics, it suffices to consider the case when $s_1 = 1$ and [1] $z_1 = 0$. We set $s = s_2$ and $z = z_2$. We let $g_{s,z}$ denote the image of $g \in G$ under our representation. When $g \in G^0$, the map $g_{s,z}$ is a piecewise translation.

There is an auxilliary homomorphism $H : G \to \text{isom}(\boldsymbol{C})$ such that $H(A_1)$ rotates [2] by $\pi/2$ radians counterclockwise around the origin and $H(A_2)$ rotates by $\pi/2$ radians counterclockwise about some other point. We usually take this other point to be 1. The homomorphism H does not depend on the parameters s and z. We say that the word $g \in G^0$ is a *statonary word* if $H(g)$ is the identity. We say that $g \in G^0$ is a *drifter* if $H(g)$ is not the identity map. The set of stationary words is the kernel of H, an infinite index subgroup of G^0.

There is a big difference between the stationary words and the drifters.

Theorem 1.2 *Suppose that $g \in G^0$ is a stationary word, and s is irrational, and z is arbitrary. Then $g_{s,z}$ has a positive density set of fixed points. In particular, $g_{s,z}$ has infinitely many periodic islands of period* 1.

Theorem 1.3 *Let $g \in G^0$ be a drifter and let s be irrational. For any $N > 0$ and any z, the map $g_{s,z}$ has only finitely many periodic islands of period less than n. Moreover, for all but countably many choices of z, the map $g_{s,z}$ has no periodic points at all.*

[1] Somewhat later on, we will find it more useful to set $z_1 = (1+i)/2$, to that the origin is the center of a square of the first grid, rather than a vertex of the grid.

[2] Note that the maps $H(A_i)$ are rigid motions of the plane. They are not directly related to the square turning maps defined above.

The word $g = (A_1A_2)^2$ is a stationary word. In this case, the results in [**S1**] give us an improved version of Theorem 1.2.

Theorem 1.4 *Let $g = (A_1A_2)^2$. Let $s \in \boldsymbol{R}_+$ be irrational and let $z \in \boldsymbol{C}$ be arbitrary. There is an unbounded sequence $\{p_k\}$ such that a positive density set of points in $\boldsymbol{C}$ are periodic with period p_k with respect to $g_{s,z}$.*

Theorem 1.4 takes a step in the direction of Conjecture 1.1. We will deduce Theorem 1.4 from some results in [**S1**], and we will give a self-contained proof for the parameter $s = \sqrt{2}/2$.

The same circle of techniques used to prove Theorem 1.4 also proves

Theorem 1.5 *Let $g = (A_1A_2)^2$. Let $s \in \boldsymbol{R}_+$ be irrational and let $z \in \boldsymbol{C}$ be arbitrary. Then for any N there are orbits having diameter greater than N. Moreover, there exists uncountably many choices of z such that $g_{s,z}$ has unbounded orbits.*

Again, we will sketch the proof of Theorem 1.5 in general, and give a self-contained proof for the parameter $s = \sqrt{2}/2$.

1.4 The Compactifications

Our proofs of the dynamical results mentioned in the previous section rely on the construction of compactifications for the square turning systems. Given $s \in \boldsymbol{R}_+$ and $z \in \boldsymbol{C}$, we get a group action of $G = (\boldsymbol{Z}/4) * (\boldsymbol{Z}/4)$ on $\boldsymbol{C}$, as discussed in the previous section. Our first result below gives a compactification of this group action.

Let $\boldsymbol{Z}[i]$ be the Gaussian integers.[3] Geometrically, $(\boldsymbol{Z}[i])^k$ is just the square grid in $\boldsymbol{C}^k$. Let $\boldsymbol{T}^4 = \boldsymbol{C}^2/(\boldsymbol{Z}[i]^2)$ be the usual 4-dimensional square torus. We define $\Psi = \Psi_s : \boldsymbol{C} \to \boldsymbol{T}^4$ by the equation

$$\Psi(x + iy) = (z, z/s) \text{ mod } (\boldsymbol{Z}[i])^2. \tag{3}$$

When s is irrational Ψ is injective on $\boldsymbol{C}$ and $\Psi(\boldsymbol{C})$ is dense in $\boldsymbol{T}^4$. Note that Ψ depends on s, but we often suppress this dependence from our notation.

[3] We often find it more convenient to work in $\boldsymbol{C}^k$ rather than $\boldsymbol{R}^{2k}$, even though sometimes we will make the identification of $\boldsymbol{C}^k$ with $\boldsymbol{R}^{2k}$.

Theorem 1.6 (Compactification) *Suppose that s is irrational. There is a piecewise affine action $\widehat{G}_s$ of G on $\boldsymbol{T}^4$ having the following properties.*

1. *For any $z \in \boldsymbol{C}$ there is some translation $\widehat{\tau}$ of $\boldsymbol{T}^4$ so that $\widehat{\tau} \circ \Psi$ conjugates the action of G_s on $\boldsymbol{C}$ to the action of $\widehat{G}_s$ on $\widehat{\tau} \circ \Psi(\boldsymbol{C})$.*

2. *For each $g \in G$, the element $\widehat{g}_s$ acts on $\boldsymbol{T}^4$ in such a way that its linear part $L_{g,s}$ independent of the point where it is evaluated.*

3. *When $g \in G^0$ is stationary, $L_{g,s}$ is the identity, so that $\widehat{g}_s$ is a polytope exchange transformation.*

4. *When $g \in G^0$ is a drifter, $L_{g,s}$ is a nontrivial complex linear parabolic. The fixed space of $L_{g,s}$ is parallel to $\Psi(\boldsymbol{C})$.*

Here is how we will deduce the dynamical results from the Compactification Theorem

- When g is stationary, the element $\widehat{g}_s$ fixes a nontrivial polytope in $\boldsymbol{T}^4$. The set $\widehat{\tau} \circ \Psi(\boldsymbol{C})$ intersects this polytope in a positive density set. This is how we will prove Theorem 1.2.

- When g is a drifter, the nontrivial parabolic nature of the action of $\widehat{g}_s$ produces a local shear which destroys the periodic points. This is how we will prove Theorem 1.3.

- When $g = (A_1A_2)^2$ the map $\widehat{g}_s$ has a compact invariant 2-dimensional slice, which we call an *octagonal PET*. In [**S1**] we showed that this invariant slice admits a renormalization scheme. This renormalization scheme produces periodic points of arbitrarily high order and also aperiodic points. We will deduce Theorems 1.4 and 1.5 by studying how $\widehat{\tau}(\boldsymbol{C})$ interacts with the orbits produced by the renormalization scheme.

Independent of the dynamical consequences, one might wonder about the geometry of the compactifications produced by the Compactification Theorem. In some cases, we can give a nice answer. In §4 we will define what we mean by a *double lattice PET*. These are special polytope exchange transformations which are defined in terms of a pair of Euclidean lattices and a pair of fundamental domains for those lattices. We first defined these maps in [**S3**], but [**S1**] has a more thorough account. The work in [**S3**] is what led us to the present paper.

Theorem 1.7 *Let $S=\{s_1,...,s_n\}$ and $Z=\{z_1,...,z_n\}$, where n is congruent to 2 mod 4. Let $R=R^2_{S,Z}$. Then there is a double lattice PET $(X,\widehat{R})$, whose domain $X\subset \boldsymbol{R}^{2n}$ is a parallelotope, and an injective map $\Psi:\boldsymbol{C}\to X$ which conjugates $(R,\boldsymbol{C})$ to the restriction of $\widehat{R}$ to $\Psi(\boldsymbol{C})$. The closure of $\Psi(\boldsymbol{C})$ is a finite invariant union of $2k$-dimensional convex polytopes, where k is the dimension of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}[s_1,...,s_n]$.*

The nicest case of our construction is when there are no rational relations amongst $s_1,...,s_n$. In this case, $\Psi(\boldsymbol{C})$ is dense in X, and $(X,\widehat{R})$ is a compactification of $(\boldsymbol{C},R)$ in the most basic sense. Near the other extreme, we can take periodic sequences $S=\{1,s,1,s,...,1,s\}$ and $Z=\{0,z,0,z,...,0,z\}$, where s is irrational. In this case, we get the following corollary, which refers to the torus action produced by the Compactification Theorem.

Corollary 1.8 *Suppose that $g\in G_0$ has the form $g=h^2$ where h has word length $n=4m+2$. Then the action of $\widehat{g}_s$ on $\boldsymbol{T}^4$ is conjugate, by a piecewise translation, to the restriction of a $(2n)$-dimensional double lattice PET to a finite invariant union of 4-dimensional convex polytopes.*

The double lattice PETs from Theorem 1.7 occur in dimensions of the form $8m+4$ for $m=0,1,2,....$ In dimension $8m+4$ there is a $4m+2$ dimensional family. [4] We will describe our examples explicitly for $m>0$ in §5, and for $m=0$ in §6.

For $m=0$, the examples have a special beauty. My monograph [**S1**] is devoted to explaining the structure associated to a 2-dimensional invariant slice of the compactification in the case $m=0$. For the purposes of explaining Theorems 1.4 and 1.5, we will re-derive some of the structure here. The case $m=0$ is also closely related to the D_4 Weyl group, though I will not discuss this here. I hope to present the details in a forthcoming paper.

1.5 Organization

This paper is organized as follows.

- In §2, we will prove the Compactification Theorem, Theorem 1.6.

[4] There is a certain redundancy, in the sense that the data $\{\lambda s_1,...,\lambda s_n\}$ leads to a system which is conjugate to the system defined by $\{s_1,...,s_n\}$. So, perhaps it is more accurate to say that there is a $4m+1$ dimensional family of examples in dimension $8m+4$.

- In §3 we prove Theorems 1.2 and 1.3.
- In §4 we prove Theorem 1.8.
- In §5 we will give an explicit description of the PETs produced by Theorem 1.8, in dimensions $4, 12, 20, 28, \ldots$.
- In §6 we consider in detail the 4-dimensional case from §5. In particular, we give accounts of Theorems 1.4 and 1.5. Most of §6 is a repeat of [**S1**] but this chapter has one new feature: I give a short and self-contained proof of the main result from [**S1**] in one nontrivial case.
- In §7 I will describe work in progress, which reveals a beautiful recursive tiling associated to the case $m = 4$ and should eventually lead to the proof of Conjecture 1.1. Details will appear in a forthcoming paper, if I can work them out.

1.6 Acknowledgements

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2 The Compactification Theorem

2.1 The Main Construction

Let $\boldsymbol{T}^{2n}$ denote the usual $2n$-dimensional torus in $\boldsymbol{C}^n$. We think of a fundamental domain for $\boldsymbol{T}^{2n}$ as the unit cube centered at the origin in $\boldsymbol{C}^n$. We let Q^{2n} denote the interior of this cube.

Given s, define

$$z = \frac{s+is}{2}. \tag{4}$$

The grid with parameters (s,z) is such that the origin the center of one of the squares. Let R_s denote the corresponding grid map.

Given a finite sequence $s_1,...,s_n$, we set $R_k = R_{s_k}$ and consider the composition

$$g = R_n \circ ... \circ R_1. \tag{5}$$

We define $\Psi : \boldsymbol{C} \to \boldsymbol{T}^{2n}$ by the map

$$\Psi(z) = \Big(\frac{z}{s_1},...,\frac{z}{s_n}\Big) \bmod (\boldsymbol{Z}[i])^n \tag{6}$$

The image $\Psi(\boldsymbol{C})$ is dense in a linear subspace of $\boldsymbol{T}^{2n}$. The dimension of this subspace, which we will discuss in more detail below depends on the number of rational relations between $s_1,...,s_n$.

We define

$$\widehat{R}_1(z_1,...,z_n) = (iz_1,z_2,...,z_n)+(-1+i)z_1\Big(0,\frac{s_1}{s_2},...,\frac{s_1}{s_n}\Big) \quad \bmod (\boldsymbol{Z}[i])^n. \tag{7}$$

This map is the identity on $\{0\}\times\boldsymbol{C}^{n-1}$ and locally affine on the set $Q^2\times\boldsymbol{T}^{2n-2}$. It is not possible to extend $\widehat{R}_1$ to all of $\boldsymbol{T}^{2n}$, but this does not bother us. The set $(\partial Q^2)\times\boldsymbol{T}^{2n-2}$, considered as a subset of $\boldsymbol{T}^{2n}$, is a union of 2 flat tori of dimension $2n-1$. This is the singular set for $\widehat{R}_1$.

We define $\widehat{R}_j$ by permuting the coordinates and interchanging the roles of s_1 with s_j. For instance,

$$\widehat{R}_2(z_1,...,z_n) = (z_1,iz_2,...,z_n)+(-1+i)z_2\Big(\frac{s_2}{s_1},0,...,\frac{s_2}{s_n}\Big) \quad \bmod (\boldsymbol{Z}[i])^n. \tag{8}$$

The map $\widehat{R}_j$ is locally affine on the product of an open square and a torus of dimension $2n-2$.

Lemma 2.1 $\widehat{R}_j \circ \Psi = \Psi \circ R_j$ *on the domain of* R_j*.*

Proof: By symmetry, it suffices to consider the case of R_1. Notice that our lemma is true for the sequence $\lambda s_1, ..., \lambda s_n$ if and only if it is true for the original sequence. For this reason, it suffices to consider the case when $s_1 = 1$. The domain of R_1 is the infinite grid of open unit squares, one of which, namely Q^2, is centered at the origin.

We first check that the equation holds in a neighborhood of 0. For $|z|$ sufficiently small, we compute

$$\Psi \circ R_1(z) = \Psi(iz) = \Big(iz, \frac{iz}{s_2}, ..., \frac{iz}{s_n}\Big). \tag{9}$$

On the other hand

$$\widehat{R}_1(\Psi(z)) = \widehat{R}_1\Big(z, \frac{z}{s_2}, ..., \frac{z}{s_n}\Big) =$$

$$\Big(iz, \frac{z}{s_2}, ..., \frac{z}{s_n}\Big) + (-1+i)z\Big(0, \frac{1}{s_2}, ..., \frac{1}{s_n}\Big) \tag{10}$$

One can see that the two expressions are equal.

Next, we observe that $\Psi(Q^2) \subset Q^2 \times \boldsymbol{T}^{2n-2}$. The restrictions of R_1 and Ψ to Q^2 are affine and the map $\widehat{R}_1$ is affine on $Q^2 \times \boldsymbol{T}^{2n-2}$. Therefore, the check we have already made implies that our lemma holds true on Q^2.

Suppose we knew that $\Psi \circ R_1(z) = \widehat{R}_1 \circ \Psi(z)$ for some Gaussian integer z. The differentials of $\Psi \circ R_1$ and $\widehat{R}_1 \circ \Psi$ at z are the same as the differentials of these maps at 0. Since they already agree at 0, they agree at z as well. This implies that our basic equation holds in a neighborhood of z. Note finally that Ψ maps the unit square Q^2_z centered at z into $Q^2 \times \boldsymbol{T}^{2n-1}$. The same continuation principle now implies that our lemma holds on Q^2_z. So, to finish the proof, we just have to check the basic equation on Gaussian integers.

When z is a Gaussian integer, we have $R_1(z) = z$. So, we just have to prove that $\widehat{R}_1$ fixes $\Psi(z)$. But $\Psi(z) \subset \{0\} \times \boldsymbol{T}^{2n-2}$, and such points are fixed by $\widehat{R}^1$. ♠

We define

$$\widehat{g} = \widehat{R}_n \circ ... \circ \widehat{R}_1. \tag{11}$$

An immediate consequence of our previous result is that

$$\Psi \circ g = \widehat{g} \circ \Psi \tag{12}$$

wherever all maps are defined.

2.2 Statement 1 of Theorem 1.6

Theorem 1.6 concerns the case $n = 2$ in the construction above. In this case, we normalize so that $s_1 = 1$ and $s_2 = s$. We will first prove Statement 1 for the choice of z in Equation 4. In this case, Equation 11 gives us the desired group action on $\boldsymbol{T}^4$. When $z = 0$, we take the translation $\widehat{\tau}$ in Theorem 1.6 to be the identity. Equation 12 gives the desired conjugacy between the two group actions. This is Statement 1.

Now we explain how things work for other choices of z. Note that the grids corresponding to the parameters z and $z + sm + isn$ are the same, for any $m, n \in \boldsymbol{Z}$. Hence, it suffices to prove Statement 1 for one representative of each point in $\boldsymbol{C}$ in the torus

$$\boldsymbol{T}^2_s = \boldsymbol{C}/\Lambda, \qquad \Lambda = \{sm + isn|\ m, n \in \boldsymbol{Z}\}. \tag{13}$$

We first show that it suffices to consider a dense set of points in the torus, and then we give an argument which covers such a dense set of points.

Lemma 2.2 *Suppose Statement 1 of the Compactification Theorem holds for a dense set of points in* $\boldsymbol{T}^2_s$. *Then Statement 1 of the Compactification Theorem holds for all points in* $\boldsymbol{C}$.

Proof: Let $z_\infty \in \boldsymbol{T}^2_s$ be some point, and let $\{z_n\}$ be a sequence of points in our dense set which converges to z_∞. Let $\widehat{\tau}_n$ be the corresponding set of translations of $\boldsymbol{T}^4$. Since the group of translations of $\boldsymbol{T}^4$ is compact, we may pass to a subsequence to that $\widehat{\tau}_n$ converges to a translation $\widehat{\tau}$. For any n, we have the equation

$$\Psi_n \circ G_{s,n} = \widehat{G}_s \circ \Psi_n, \qquad \Psi_n = \widehat{\tau}_n \circ \Psi. \tag{14}$$

Here $G_{s,n}$ is the group action corresponding to z_n. Since Ψ_n is an injection, Equation 14 is equivalent to Statement 1 of the Compactification Theorem for the parameter z_n.

Consider what happens as $n \to \infty$. For any given element $g \in G$, the maps defined by Equation 14 converge uniformly on compact sets to the corresponding maps defined in terms of the limit parameter z. Hence, Equation 14 holds as well for the $n = \infty$. But then Statement 1 of the Compactification Theorem holds for z_∞. ♠

Let z_{00} be as in Equation 4. Consider the points

$$z_{mn} = z_{00} + m + in, \qquad m, n \in \boldsymbol{Z}. \tag{15}$$

What makes these points special is that the group action defined in terms of z_{mn} is conjugate to the group action defined in terms of z_{00}. It is not generally true that the group actions defined in terms of different choices of z are conjugate.

When s is irrational, the set of representatives of the points $\{z_{mn}\}$ in $\boldsymbol{T}_s^2$ is dense. So, it suffices to prove Statement 1 of the Compactification Theorem for the points z_{mn}.

Lemma 2.3 *Statement 1 of the Compactification Theorem holds for z_{mn}.*

Proof: Let $G_{s,m,n}$ be the group action corresponding to the point z_{mn}. Let τ_{mn} be the translation of $\boldsymbol{C}$ which carries z_{00} to z_{mn}. By construction, τ_{mn} conjugates $G_{s,0,0}$ to $G_{s,m,n}$. More specifically,

$$\tau \circ G_{s,m,n} \circ \tau^{-1} = G_{s,0,0}. \tag{16}$$

Since Ψ is locally affine, there is some translation $\widehat{\tau}_{mn}$ such that

$$\Psi \circ \tau_{mn} = \widehat{\tau}_{mn} \circ \Psi. \tag{17}$$

For ease of notation we make the following abbreviations.

$$\tau = \tau_{mn}, \qquad \widehat{\tau} = \widehat{\tau}_{mn}, \qquad G = G_{s,0,0}, \qquad G' = G_{s,m,n}, \qquad \widehat{G} = \widehat{G}_s. \tag{18}$$

Combining Equations 16 and 18 with the special case of Statement 1 we have already proved, we have

$$\widehat{\tau}\Psi G'\tau^{-1} = \Psi\tau G'\tau^{-1} = \Psi G = \widehat{G}\Psi. \tag{19}$$

Hence

$$\widehat{\tau}\Psi G' = \widehat{G}\Psi\tau = \widehat{G}\widehat{\tau}\Psi. \tag{20}$$

This last equation says that $\widehat{\tau}_{m,n} \circ \Psi$ is a semi-conjugacy between $G_{s,m,n}$ and $\widehat{G}_s$. This equivalent to Statement 1 for z_{mn}. ♠

2.3 The Rest of Theorem 1.6

Statement 2 of Theorem 1.6 is immediate. The linear parts of the maps $\widehat{R}_1$ and $\widehat{R}_2$ are independent of the point where they are evaluated, and so the same goes for any word in these generators.

Statements 3 and 4 of the Compactification Theorem are purely statements of linear algebra. The linear parts of $\widehat{R}_1$ and $\widehat{R}_2$ are given by

$$L_1 = \begin{bmatrix} i & 0 \\ -1/s + i/s & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & -s + is \\ 0 & i \end{bmatrix} \tag{21}$$

We introduce the matrix

$$E = \begin{bmatrix} s & 0 \\ 1 & 1 \end{bmatrix}. \tag{22}$$

E is a basis of eigenvectors of L_1. Setting $M_k = E^{-1} L_k E$, we have

$$M_1 = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} i & -1 + i \\ 0 & 1 \end{bmatrix}. \tag{23}$$

As long as $s \neq 0$, the matrix E is nonsingular. M_1 and M_2 do not depend on s, so the conjugacy class of some word in L_1 and L_2 does not depend of s.

A calculation shows that M_1 and M_2 both preserve $\Pi = \boldsymbol{C} \times \{-1\}$. Moreover, M_1 rotates Π by $\pi/2$ counterclockwise about the point $(0, -1)$ and M_2 does the same thing about the point $(1, -1)$. Therefore, the auxilliary homomorphism H discussed in connection with Theorem 1.6 can be interpreted as the map which carries A_k (a generator of $G = \boldsymbol{Z}/4 * \boldsymbol{Z}/4$) to $M_k|\Pi$ for $k = 1, 2$. In short $H(A_k) = M_k|\Pi$ for $k = 1, 2$. So, g is a stationary word if and only if the corresponding word in M_1 and M_2 acts trivially on Π.

More formally, let $g \in G^0$. Let μ_g be the word in M_1 and M_2 corresponding to g. By construction, $E^{-1} L_g E = \mu_g$. Here L_g is the linear part of $\widehat{g}$. Note that $g \in G_0$ forces μ_g to be the identity on $\boldsymbol{C} \times \{0\}$. Suppose $H(g)$ is trivial. Then μ_g is the identity on both $\boldsymbol{C} \times \{0\}$ and $\boldsymbol{C} \times \{-1\}$. Since μ_g is complex linear, this forces μ_g to be the identity. Hence L_g is the identity as well. This proves Statement 3.

We use the same notation for Statement 4. If $H(g)$ is not the identity, then the nontrivial action of μ_g on Π forces μ_g to be a nontrivial parabolic. Since μ_g is complex linear and acts as the identity on $\boldsymbol{C} \times \{0\}$, it must be the case that μ_g is a parabolic of real rank 2, when interpred as acting on $\boldsymbol{R}^4$. The same goes for L_g, which is conjugate to μ_g over $\boldsymbol{C}$ and *a forteriori* over $\boldsymbol{R}$. This proves Statement 4.

3 Existence of Periodic Points

3.1 Good Partitions

We fix some drifter $g \in G^0$ and consider $\widehat{g}_s$ acting on $\boldsymbol{T}^4$. Recall that the unit cube Q^4 is our fundamental domain for $\boldsymbol{T}^4$. Recall also that $\widehat{g}_s$ has an invariant foliation $F = F_s$ that is parallel to $\Phi_s(\boldsymbol{C})$.

We say that a a *small polytope* is a convex polytope $P \subset Q^4$ such that no face of P contains an open subset of a plane in F. We also insist that $g(P) \subset Q^4$. In other words, neither P nor $g(P)$ is allowed to cross the boundary of Q^4. Given a piecewise affine map $h : \boldsymbol{T}^4 \to \boldsymbol{T}^4$, we say that a partition Π of $\boldsymbol{T}^4$ is *good* for h if Π consists of small polytopes and h is defined and affine on the interior of each one.

Lemma 3.1 *There is a good partition for $\widehat{g}_s$.*

Proof: We suppress the dependence on s from our argument. From the construction in §2.1, we see that $\widehat{R}_1$ and $\widehat{R}_2$ are defined on the complement of a finite union of flat 3-tori. These 3-tori are transverse to our foliation. By removing additional 3-dimensional flats, including the boundary of the unit cube, we can find a partition which is simultaneously good for all the words $\widehat{R}_i^j$ where $i = 1, 2$ and $j = \pm 1$.

We produce a good partition for any word in $\widehat{R}_1$ and $\widehat{R}_2$ by induction on the length of the word. Suppose that $g = \widehat{R}_1 h$ and $\Pi_h = \{H_1, ..., H_n\}$ is a good partition for h. Let $\{A_1, ..., A_m\}$ be a good partition for $\widehat{R}_1$. We define

$$G_k = \bigcup_{i=1}^{m} h^{-1}(h(H_k) \cap A_i) \qquad k = 1, ..., n. \tag{24}$$

Then G_k is a partition of H_k into finitely many small polytopes such that g is defined on each one. If B is some polytope in this partition, then $g(B) \subset \widehat{R}_1(A_i)$ for some i. Hence $g(B)$ is also small. The union of all the polyhedra in G_k as k ranges from 1 to n, gives the desired partition for g. A similar construction works for $\widehat{R}_i^j h$ for the other relevant choices of i and j. ♠

For use in the next chapter, we record the following corollary of our construction above.

Corollary 3.2 *For any n, there is a good partition Π_n for $\widehat{g}_s^n$. Moreover we can choose these partitions so that Π_n is a refinement of Π_m when $n > m$.*

3.2 Proof of Theorem 1.3

Let $g \in G^0$ be a drifter. We first show that $\widehat{g}_{s,z}$ has only finitely many periodic islands of period less than N. Since there are only finitely many positive integers less than N, it suffices to prove instead that $\widehat{g}_{s,z}$ has only finitely many fixed points of period N. Replacing $\widehat{g}_{s,z}$ by $g^N_{s,z}$, it suffices to prove the result when $N = 1$.

That is, we want to prove that $g_{s,z}$ has only finitely many fixed islands. By *fixed island*, we mean a periodic island corresponding to a point of period 1. Any two fixed islands are either identical or have disjoint interiors. We will prove the result for $z = (s + is)/2$, as in Equation 4. The general case has the same proof, except that the map $\widehat{\tau} \circ \Phi$ is used in place of the map Φ, for some suitable translation $\widehat{\tau}$. For ease of notation we set $g = \widehat{g}_{s,z}$.

Let Π be a good partition for $\widehat{g}$. By the Compactification Theorem, $\widehat{g}$ preserves F and the restriction of $\widehat{g}$ to a suitable leaf of the foliation F is conjugate to the action of g on $\boldsymbol{C}$.

Say that a *good disk* is an open polygon of the form $D = X \cap P$, where X is a 2-plane parallel to the foliation F. We say that D is *fixed* if $\widehat{g}$ fixes D pointwise.

Lemma 3.3 *If D is a good disk but not a fixed good disk, then $\widehat{g}$ has no fixed points in D.*

Proof: Recall that D is parallel to the eigenspace of the linear part of $\widehat{g}$, and the eigenvalues are all 1. Hence, the restriction of g to D is a translation. This means that D is either a fixed disk or g fixes no points of D at all. ♠

Now we come to the key structural result.

Lemma 3.4 *Each small polytope of Π contains at most one fixed good disk. Hence, the set of fixed points of $\widehat{g}$ is contained in finitely many good disks.*

Proof: Suppose that $\widehat{g}$ fixes two good disks in a good polytope P. The restriction of $\widehat{g}$ to the interior of P is affine. Let A be the affine map which extends $\widehat{g}|P$. By construction, A is the identity on two parallel 2-planes. But then A is the identity on a certain 3-dimensional subspace. But this is impossible, because the linear part of A is a parabolic of real rank 2. ♠

Let X denote the set of fixed points of g. Let $X' \subset X$ denote these points x such that $\Phi(x)$ lies in the interior of some small polytope of the good partition. Since the faces of these small polytopes are transverse to the invariant foliation, we see that X' is dense in X. In particular, every fixed island contains points of X'.

Let Y denote the set of fixed good disks. Since $\Psi_s : \boldsymbol{C} \to \boldsymbol{T}^4$ is injective and since all the fixed points of $\widehat{g}$ lie in fixed good disks, Ψ_s induces a map from X' into Y.

Lemma 3.5 *Suppose that $p, q \in X'$. If Ψ maps p and q into the same fixed good disk, then p and q lie in the same fixed island.*

Proof: Let I_p and I_q be the fixed islands containing p and q respectively. Either $I_p = I_q$ or these sets have disjoint interiors. If $\Psi(p)$ and $\Psi(q)$ lie in the same good fixed disk D, then Ψ maps the line segment $\overline{pq}$ into D. But then $\widehat{g}$ fixes every point of $\Psi(\overline{pq})$. Hence $\widehat{g}$ is entirely defined, and the identity, on $\overline{pq}$. But then $\overline{pq} \subset I_p \cap I_q$. This forces $I_p = I_q$. ♠

If g had infinitely many fixed islands, then the previous result would give us infinitely many fixed good disks. This is a contradiction. Hence g only has finitely many fixed islands. As we mentioned above, this completes the proof that, for any N, the map $g = g_{s,z}$ has only finitely many periodic islands of period less than N. This is the first statement of Theorem 1.3.

Now we prove Statement 2 of Theorem 1.3, which says that $g_{s,z}$ has periodic points only for countably many choices of z. Using the same trick as for Statement 1, it suffices to prove this result for fixed points.

Let $\widehat{\tau}_z$ be the translation of $\boldsymbol{T}^4$ such that $\widehat{\tau} \circ \Psi_s$ conjugates the action of $g_{s,z}$ on $\boldsymbol{C}$ to the action of $\widehat{g}_s$ on

$$\boldsymbol{C}_z = \widehat{\tau}_z \circ \Psi_s(\boldsymbol{C}). \tag{25}$$

Here $\boldsymbol{C}_z \subset \boldsymbol{T}^4$ is one of the leaves of the invariant foliation. Note that the map $\boldsymbol{C} \to \boldsymbol{C}_z$ is a bijection. Here is the key lemma for our result.

Lemma 3.6 *If $g_{s,z}$ has a fixed point, then $\boldsymbol{C}_z$ is one of finitely many leaves of the invariant foliation F.*

Proof: There are only finitely many fixed good disks. Hence there are only finitely many leaves of F which contain good fixed disks. Call these leaves *special*. If $g_{s,z}$ has a fixed point then $\boldsymbol{C}_z$ contains an fixed good disk and hence is special. ♠

Again, we call a leaf in the invariant foliation special if it contains a fixed good disk. The next result shows that there are only countable many choices of w such that $\boldsymbol{C}_w$ is a special leaf. This finishes the proof that there are only countably many choices of z for which $g_{s,z}$ has a fixed point.

Lemma 3.7 *For each $z \in \boldsymbol{C}$, there are only countably many choices of points $w \in \boldsymbol{C}$ such that $\boldsymbol{C}_z = \boldsymbol{C}_w$.*

Proof: Suppose that $\boldsymbol{C}_z = \boldsymbol{C}_w$. We can interpret the translation by $\widehat{\tau}_z - \widehat{\tau}_w$ as a translation of $\boldsymbol{C}$ which conjugates the action of of $G_{s,z}$ to the action of $G_{s,w}$. Call this translation τ.

Since τ conjugates the rotation R_1 to the rotation R_1, we see that τ must be translation by some integer lattice vector. In particular, τ preserves the grid G_1. At the same time, $\tau(G_{s,z}) = G_{s,w}$. Hence, there are only countably many choices for $G_{s,w}$. But only countably many choices of w lead to the same choice of $G_{s,w}$. Hence, there are only countable many choices of w which lead to one of the countably many possible choices of grid. ♠

3.3 Proof of Theorem 1.2

Now suppose that $g \in G^0$ is a stationary word. Inspecting the maps $\widehat{R}_1$ and $\widehat{R}_2$ which define the group action $\widehat{G}_s$, we see that both elements fix the origin $O_4 \in \boldsymbol{T}^4$ and both elements are defined in a neighborhood of O_4. Hence, for any $g \in G$, the word $\widehat{g}_s$ fixes O_4 and is defined in a neighborhood of O_4. When $g \in G^0$ is a stationary word, $\widehat{g}_s$ is a piecewise translation. In this case, there is some nontrivial and maximal open polytope P which contains O_4 in its interior such that $\widehat{g}_s$ fixes every point of P.

We choose $z \in \boldsymbol{C}$ and let $\Psi = \tau \circ \Psi_s$ be the map from the Compactification Theorem. Let $P' = \Psi^{-1}(P)$. Since Ψ is locally affine and $\Psi(\boldsymbol{C})$ is dense in $\boldsymbol{T}^4$, we see that P' has positive density. In fact, the density of P' is just the volume of P. But $g_{s,z}$ fixes every point of P'. Hence $g_{s,z}$ has a positive density set of fixed points. Since P' has positive density, P' is unbounded. Since every fixed island is compact, there must be infinitely many fixed islands. This completes the proof of Theorem 1.2.

4 Nature of the Compactifications

4.1 Double Lattice PETs

In this chapter we prove Theorem 1.7. For starters, we define what we mean by a double lattice PET.

The data for a double lattice PET is a quadruple $(X_1, X_2, \Lambda_1, \Lambda_2)$, where

- X_1 and X_2 are polytopes in $\boldsymbol{R}^n$.
- Λ_1 and Λ_2 are lattices in $\boldsymbol{R}^n$.
- X_i is a fundamental domain for Λ_j for all possible $i, j \in \{1, 2\}$.

To say that Λ is a lattice is to say that there is some affine isomorphism T of $\boldsymbol{R}^n$ such that $\Lambda = T(\boldsymbol{Z}^n)$. To say that X is a fundamental domain for Λ is to say that the orbit $\Lambda(X)$ tiles $\boldsymbol{R}^n$: The translates have pairwise disjoint interiors and the union of the translates is a covering.

In all our examples, the polytopes X_1 and X_2 are parallelotopes centered at the origin.

There is a natural map $f_{ij} : X_i \to X_j$ defined as follows. Given $p \in X_i$, there is generically a unique vector $\lambda_p \in \Lambda_j$ such that $p + \lambda_p \in \Lambda_j$. We define

$$f_{ij}(p) = p + \lambda_p. \tag{26}$$

The maps f_{11} and f_{22} are the identity; we do not care about these maps. The maps f_{12} and f_{21} are the ones of interest to us. These maps are piecewise translations. The composition

$$f = f_{21} \circ f_{12} : X_1 \to X_1 \tag{27}$$

is a polytope exchange transformation (PET) having X_1 as a domain. We call the system (X_1, f) a *double lattice PET*.

We needed to break symmetry in order to define the maps f_{ij}. We could equally well define the maps $g_{ij} : X_i \to X_j$ as follows. For $p \in X_i$ there is generically a unique vector $\mu_p \in \Lambda_i$ such that $p + \mu_i \in X_j$. We can then define $g_{ij}(p) = p + \mu_p$. It is easy to see that $g_{ij} = f_{ji}^{-1}$. Hence $f^{-1} = g_{12}g_{21}$.

At first it might seem difficult to produce quadruples satiasfying the necessary conditions. However, in [**S3**] we showed that essentially all polygonal outer billiards systems have compactifications which are double lattice PETs. In this chapter we will see that the construction in the previous chapter leads naturally to double lattice PETs as well.

4.2 The Linear Part

We continue with the notation from §2.1 As a start on the proof of Theorem 1.7, we analyze the linear part L_g of $\widehat{g}$ in case g is a word of length $n = 4m+2$. Here is our main result.

Lemma 4.1 *L_g is an involution whose (-1)-eigenspace is 2-dimensional and whose $(+1)$-eigenspace is $(2n-2)$-dimensional.*

We prove Lemma 4.1 through a series of smaller results. We also note that the calculations done in §5.4 give an alternate proof of Lemma 4.1.

Lemma 4.2 *The (-1) eigenspace of L_g has real dimension at least 2.*

Proof: The action of $\widehat{g}$ on $\Psi(\boldsymbol{C})$ is conjugate to the action of g on $\boldsymbol{C}$. Since g has length $4n_2$, the linear part of g is rotation by π. Hence, the same goes for $\widehat{g}$. This implies that L_g preserves the complex line through the origin and parallel to $\Psi(\boldsymbol{C})$. In other words, considered as a real matrix L_g has a (-1)-eigenspace which is at least 2 dimensional. ♠

To finish the proof of Lemma 4.1, we will produce a $(2n-2)$-dimensional subspace on which L_g is the identity. This will finish the proof. We will first consider the case when the numbers $1, s_2, ..., s_n$ have no rational relations amongst them. Once we take care of this case, we will deduce the general case by a limiting argument. When there are no rational relations amongst the numbers $1, s_2, ..., s_n$, the image $\Psi(\boldsymbol{C})$ is dense in $\boldsymbol{T}^{2n}$.

We say that a subset $\Sigma \subset \boldsymbol{C}$ is a *net* if there is some K such that every point of $\boldsymbol{C}$ is within K of some point of Σ. If U is any open subset of $\boldsymbol{T}^{2n}$, the inverse image $\Psi^{-1}(U)$ is a net in $\boldsymbol{C}$.

Lemma 4.3 *Let $\epsilon > 0$ be given. If U is sufficiently small then every point of $\Psi^{-1}(U)$ is within ϵ of a fixed point of g.*

Proof: We will crucually use the fact that the linear part of g is rotation by π. Consider a point $z \in \boldsymbol{C}$ such that $\Psi(z)$ lies within δ of the origin. Then z is very nearly the center of all the grids used to define the maps $R_1, ..., R_n$. Were z the center of all these grids, g would locally be a rotation by π about z. As it is, g is a small perturbation of a rotation by π about z, and the differential dg is still rotation by π. But, in this situation, g has a fixed point z' very close to z. The distance $|z-z'|$ only depends on δ and n. ♠

Corollary 4.4 *Let U be any open set containing the origin in $\boldsymbol{C}^n$. The set of fixed points of g mapping into U is a net in $\boldsymbol{C}$.*

We choose some small open set U as in the corollary, and let H denote the smallest linear subspace containing all the points of the set $\Psi(\Sigma)$. By construction, $\widehat{g}$ fixes every point of $\Psi(\Sigma)$. Since these points span H, we see that $\widehat{g}$ is the identity on H. We just have to prove that H has dimension $2n-2$.

Lemma 4.5 *H has dimension at least $2n-2$.*

Proof: Suppose that H has dimension $d < 2n-2$. Given any point $p \in \Sigma$, there is some small disk Δ so that the restriction of g to Δ is rotation by π. The radius of Δ can be chosen to be uniformly large. Call this radius ρ. The image $\Psi(\Delta)$ is an isometric disk centered at a point of H and parallel to $\Psi(\boldsymbol{C})$.

Let Σ' denote the ρ-tubular neighborhood of Σ. By construction, $\Psi(\Sigma')$ is a union of disks, all parallel to $\Psi(\boldsymbol{C})$, and all centered at points of H. But this implies that the closure of K of $\Psi(\Sigma')$ has dimension at most $d+2 < 2n$.

On the other hand, since Σ is a net, Σ' is a set of positive density in $\boldsymbol{C}$. Given the affine nature of Ψ and the fact that $\Psi(\boldsymbol{C})$ is dense in $\boldsymbol{T}^n$, we see that K must have positive volume – i.e. $(2n)$-dimensional Lebesgue measure. This is impossible if $\dim(K) < 2n$. This contradiction shows that $\dim(H) \geq 2n-2$. ♠

We have produced a subspace of dimension at least $2n-2$ which is fixed by $\widehat{g}$. This means that the linear part L_g has a $(+1)$-eigenspace of dimension at least $2n-2$. This is all we needed for Lemma 4.1 in the arational case.

In case there are some rational relations between the numbers $1, s_2, .., s_n$, we can perturb the numbers slightly to get a new sequence with no rational relations. Hence, in the general case, the linear part of $\widehat{g}$ has a sequence of approximations by linear maps which all have $(+1)$-eigenspaces of dimension $2n-2$. But then the dimension of the $(+1)$-eigenspace of the limiting map must be at least $2n-2$. This completes the proof of Lemma 4.1 in the general case.

We mention the obvious corollary of Lemma 4.1

Corollary 4.6 *Suppose that $g = h^2$ where h is a word of length $4m+2$. Then the linear part of $\widehat{g}$ is the identity. Hence $(\boldsymbol{T}^{2n}, \widehat{g})$ is a PET.*

4.3 A Picture

Before we continue our analysis in the next section, we show a picture that the reader should keep in mind throughout the discussion.

The square shown in Figure 4.1 is meant to be the torus $\boldsymbol{T}^2$. The opposite sides are meant to be identified. The thick line Ω_1 is really a circle and the thick line Ω_2 is really a line segment. The set $\Omega_1 \cup \Omega_2$ is meant to be a toy version of the singular sets for the map $\widehat{h}$ we consider in the next section. The complement of the singular set is a parallelogram – i.e. a set affinely eqiovalent to Q^2 – embedded in $\boldsymbol{T}^2$ in a funny way.

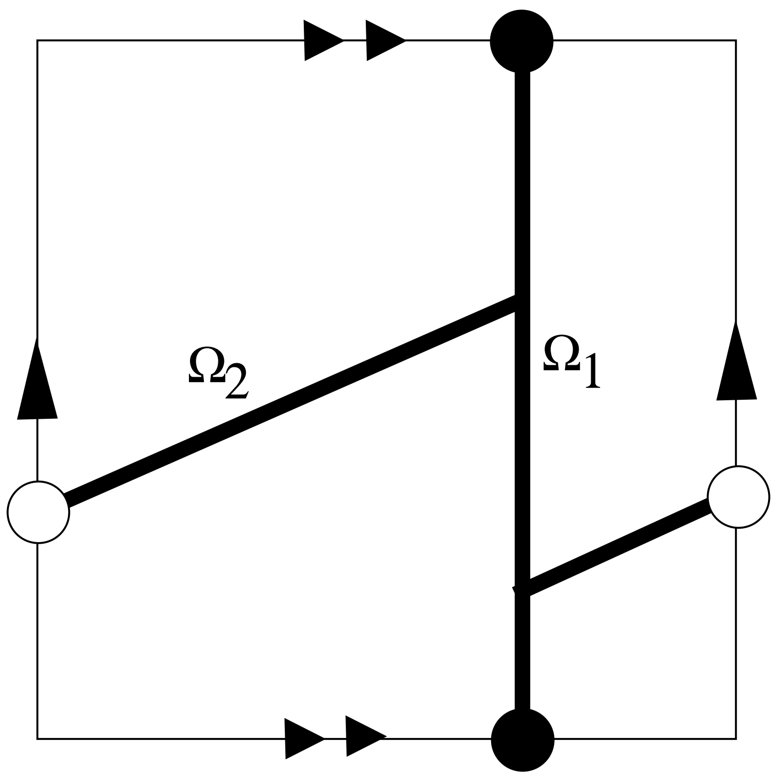

Figure 4.1: A parallelogram sitting inside a torus.

The set Ω_1 is a 1-torus sitting inside $\boldsymbol{T}^2$. The set Ω_2 is obtained from a horizontal line by applying a locally affine map defined in the annulus $D_1 = \boldsymbol{T}^2 - \Omega_1$. This locally affine map does not extend to Ω_1; it is a kind of partial Dehn twist. The universal cover Δ_1 of D_1 is an infinite strip. This strip is a low dimensional version of the slabs Δ_k we consider in our proof below.

4.4 The Singular Set

We continue with the case where $g = h^2$ and h has length $n = 4m + 2$. Let $\Sigma_j \subset \boldsymbol{T}^{2n}$ denote the singular set of $\widehat{R}_j$. Recall that Σ_j is a union of 2 codimension 1 tori. Let $\Omega_1 = \Sigma_1$ and then

$$\Omega_k = \widehat{R}_1^{-1} \circ ... \circ \widehat{R}_{k-1}^{-1}(\Sigma_k), \qquad k = 2, ..., n. \tag{28}$$

It follows from induction that $\widehat{h}$ is entirely defined on

$$\boldsymbol{T}^{2n} - \Omega, \qquad \Omega = \bigcup_{j=1}^{n} \Omega_j. \tag{29}$$

Recall that Q^k is the open unit k-dimensional cube centered at the origin. The goal of this section is to prove the following result.

Lemma 4.7 *$\boldsymbol{T}^{2n} - \Omega$ is affinely equivalent to Q^{2n}.*

We will prove Lemma 4.7 through a series of smaller results.

Say that a *singular hyperplane* is a hyperplane parallel to some Ω_j. Each Ω_j supplies 2 singular hyperplanes, so there are $2n$ singular hyperplanes in total.

Lemma 4.8 *The $2n$ singular hyperplanes are linearly independent in the sense that their normal vectors form a real basis for $\boldsymbol{R}^{2n}$.*

Proof: We use complex notation. Let $e_1, ..., e_6$ denote the standard basis vectors for $\boldsymbol{C}^6$. Let L_j denote the linear part of $\widehat{R}_j$. The hyperplanes associated to Ω_1 are

$$(e_1)^\perp,\ (ie_1)^\perp,\ L_1^{-1}((e_2)^\perp),\ L_1^{-1}((ie_2)^\perp),\ L_1^{-1}L_2^{-1}((e_3)^\perp),\ L_1^{-1}L_2^{-1}((ie_3)^\perp), ... \tag{30}$$

The corresponding normals are given by

$$e_1, \quad ie_1, \quad L_1^t(e_2), \quad L_1^t(ie_2), \quad L_1^tL_2^t(e_3), \quad L_1^tL_2^t(ie_3), ... \tag{31}$$

An easy calculation shows that the kth column of $L_1^t...L_{k-1}^t$ has a 1 in the (kk)th position and (0)s below. This implies that the normals in Equation 31 are linearly independent. ♠

We will prove Lemma 4.7 by induction. Define

$$D_k = \boldsymbol{T}^{2n} - (\Omega_1 \cup ... \cup \Omega_k). \tag{32}$$

Our final goal is to show that D_n is affinely equivalent to Q^{2n}. Our induction step will be that D_k is affinely equivalent to $\boldsymbol{Q}^{2k} \times \boldsymbol{C}^{n-k}$. We have already seen that this statement is true for $k = 1$. We will suppose that the statement is true for some choice of k and then prove it for $k+1$.

Define

$$\widehat{h}_k = \widehat{R}_k \circ ... \circ \widehat{R}_1. \tag{33}$$

The map $\widehat{h}_k$ is defined on D_k. We have projection

$$\pi : \boldsymbol{C}^n \to \boldsymbol{T}^{2n}. \tag{34}$$

Let L_k be the linear part of $\widehat{h}_k$. Let $\Delta_k \subset \boldsymbol{C}^n$ denote the universal cover of D_k. We think of Δ_k as being one connected component of the preimage $\pi^{-1}(D_k)$. The set Δ_k is affinely equivalent to the "slab" $Q^{2k} \times \boldsymbol{C}^{n-k}$.

We have a commuting square

$$\begin{array}{ccc} \Delta_k & \stackrel{L_k}{\to} & \boldsymbol{C}^n \\ \downarrow \pi & & \downarrow \pi \\ D_k & \stackrel{\widehat{h}_k}{\to} & \boldsymbol{T}^{2n} \end{array} \tag{35}$$

The set $\pi^{-1}(\Sigma_{k+1})$ consists of two infinite families of parallel hyperplanes. We call these infinite families A_{k+1} and B_{k+1}. The two families are transverse to each other. By Lemma 4.8, the hyperplanes in $L_k^{-1}(A_{k+1})$ and $L_k^{-1}(B_{k+1})$ are transverse to $\partial\Delta_k$.

The intersection $\Delta_k \cap L_k^{-1}(A_{k+1})$ is an infinite parallel family of sets, each of which is affinely equivalent to $Q^{2k} \times \boldsymbol{C}^{n-k-1}$. The same goes for $D_k \cap L_k^{-1}(B_{k+1})$. The projection π carries these sets to Ω_{k+1}.

Let Δ_{k+1} be some connected component of

$$\Delta_k - L_k^{-1}(A \cup B).$$

The set Δ_{k+1} is affinely equivalent to $\boldsymbol{Q}^{2k+2} \times \boldsymbol{C}^{2n-2k-2}$. Informally, the sets $L_k^{-1}(A_{k+1})$ and $L_k^{-1}(B_{k+1})$ chop up two of the noncompact directions into compact pieces. The map $\pi : \Delta_{k+1} \to D_{k+1}$ is the universal covering map. From this picture we see that D_{k+1} is affinely equivalent to $\boldsymbol{Q}^{2k+2} \times \boldsymbol{T}^{2n-2k-2}$. This completes the induction step.

4.5 The Double Lattice PET

Now we start enhancing the notation from the previous section. Define

$$\Lambda_1 = (\boldsymbol{C}[i])^n. \tag{36}$$

Let $\pi : \boldsymbol{C}^n \to \boldsymbol{C}^n/\Lambda_1$ be projection, as above.

Since $\boldsymbol{T}^{2n} - \Omega$ is affinely equivalent to the contractible set Q^{2n}, we have a parallelotope X_1 and a continuous local inverse

$$\pi^{-1} : \boldsymbol{T}^{2n} - \Omega \to X_1 \subset \boldsymbol{C}^n. \tag{37}$$

The parallelotope X_1 is just the image of $\boldsymbol{T}^{2n} - \Omega$ under the lift π^{-1}. Since π is injective on X_1 and $\pi(X_1)$ has full measure in $\boldsymbol{T}^{2n}$, we see that X_1 is a fundamental domain for Λ_1.

Lemma 4.9 *$\widehat{h} = \pi \circ L_h \circ \pi^{-1}$ on $\boldsymbol{T}^{2n} - \Omega$.*

Proof: Both maps agree in a neighborhood of the origin and are locally affine and entirely defined on the contractible domain in question. Hence, these two maps agree everywhere. ♠

Now we introduce another lattice and another parallelotope. Recall that the linear part L_h of $\widehat{h}$ is an affine involution. We define

$$X_2 = L_h(X_1), \qquad \Lambda_2 = L_h(\Lambda_1). \tag{38}$$

Now we have specified all the data for a double lattice PET. We just have to check the conditions.

Lemma 4.10 *X_2 is a fundamental domain for Λ_1.*

Proof: We already know that $\widehat{h} = \pi \circ L_h \circ \pi^{-1}$ on $T^{2n} - \Omega$. But this means that the map $\pi \circ L_h : X_1 \to \boldsymbol{T}^{2n}$ is injective and has dense image. But this means that $\pi : L_h(X_1) \to \boldsymbol{T}^{2n}$ is injective and has dense image. Hence $L_h(X_1)$ is a fundamental domain for Λ_1. But $L_h(X_1) = X_2$. ♠

Since L_h is an involution, we see that X_1 is also a fundamental domain for Λ_2. In short, the data $(X_1, X_1, \Lambda_1, \Lambda_2)$ define a double lattice PET. We denote this PET by (X, f). Here $X = X_1$ is the domain.

Lemma 4.11 *The locally affine isomorphism* $\pi : X_1 \to \boldsymbol{T}^{2n} - \Omega$ *conjugates the system* (X, f) *to the system* $(\boldsymbol{T}^{2n}, \widehat{g})$.

Proof: We first want to understand the map

$$F = \pi^{-1} \circ \pi : X_2 \to X_1.$$

For any $p \in X_2$, the points p and $F(p)$ differ by an element of Λ_1. Thus, the map $F_1 : X_2 \to X_1$ is just the map f_{21} discussed in §4.1. In short,

$$f_{21} = \pi^{-1} \circ \pi. \tag{39}$$

Since L_h is an involution which interchanges the roles of X_1 and X_2, and also the roles of Λ_1 and Λ_2, we have

$$f_{12} = L_h \circ \pi^{-1} \circ \pi \circ L_h. \tag{40}$$

Now we put these equations together.

$$\begin{aligned} f &= \\ f_{21} \circ f_{12} &= \\ \pi^{-1} \circ \pi \circ L_h \circ \pi^{-1} \circ \pi \circ L_h &=_1 \\ \pi^{-1} \circ \pi \circ L_h \circ \pi^{-1} \circ \pi \circ L_h \circ (\pi^{-1} \circ \pi) &= \\ \pi^{-1} \circ (\pi \circ L_h \circ \pi^{-1}) \circ (\pi \circ L_h \circ (\pi^{-1}) \circ \pi &=_2 \\ \pi^{-1} \circ \widehat{h} \circ \widehat{h} \circ \pi &= \\ \pi^{-1} \circ \widehat{g} \circ \pi. \end{aligned} \tag{41}$$

Equality 1 comes from the fact that $\pi^{-1} \circ \pi$ is the identity on $X = X_1$. Equality 2 is two applications of Lemma 4.9. In short, $f = \pi^{-1} \circ \widehat{g} \circ \pi$. ♠

It is worth emphasizing that the domain for π^{-1} is $\boldsymbol{T}^{2n} - \Omega$, though we think of π as giving a piecewise isometric conjugacy between a system in $\boldsymbol{T}^{2n}$ and a system in X_1. Since our maps are not everywhere defined, the difference in topology has no meaning here.

4.6 The Invariant Slice

Consider the composition

$$\Xi = \pi^{-1} \circ \Psi : \boldsymbol{C} \to X_1. \tag{42}$$

Ξ is not defined on the set $\Psi^{-1}(\Omega)$. This set is a countable collection of line segments. The fact that Ξ is not defined at these points does not bother us. These are the points where the original planar system $g_{s,z}$ is not defined. Putting together the results above, we see that Ξ conjugates the action of $g_{s,z}$ on $\boldsymbol{C}$ to the action of the double lattice PET (X, f) on $\Xi(\boldsymbol{C})$.

Now let k denote the dimension of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}[s_1, ..., s_n]$. Looking at the map Ψ, we see that $\Psi(\boldsymbol{C})$ is a $(2k)$-dimensional linear subspace of $\boldsymbol{T}^{2m}$. The intersection

$$\widehat{C} = \text{closure}(\Psi(\boldsymbol{C})) \cap (\boldsymbol{T}^{2n} - \Omega) \tag{43}$$

is a finite invariant union of $(2k)$-dimensional open polytopes. The set $\pi^{-1}(\widehat{C})$ is likewise an invariant finite union of convex $(2k)$-dimensional polytopes in X_1. At the same time, this set is the closure of $\Xi(\boldsymbol{C})$ in X_1. This proves Theorem 1.7.

4.7 Proof of Corollary 1.8

It only remains to reconcile our compactification here with the one from Theorem 1.6. recall that $n = 4m + 2$. In case we have the sequence $1, s, 1, s, ..., 1, s$, the construction here is identical to the construction given in §2, except that we are repeating the coordinates $2m + 1$ times. In other words, the diagonal embedding $\boldsymbol{C}^2 \to \boldsymbol{C}^m$ carries the compactification produced by Theorem 1.6 to the system $(\widehat{T}^{2n}, \widehat{g})$. The composition of π^{-1} with the diagonal embedding gives the desired conjugacy.

5 A Concrete Family of PETs

5.1 Generalities

We will work in $\boldsymbol{C}^n$. We can specify [5] a double lattice PET $(X_1, X_2, \Lambda_1, \Lambda_2)$ by a quadruple of $n \times n$ matrices $(\chi_1, \chi_2, L_1, L_2)$, where

- $X_j = \chi_j(Q^{2n})$.
- Λ_j is the $\boldsymbol{Z}[i]$-span of the columns of L_j.

Here $Q^{2n} = (Q^2)^n$, where Q^2 is the unit square centered at the origin in $\boldsymbol{C}$.

There is some ambiguity in our choice of matrices. Let us call two vectors V and V' equivalent if either $V' = \omega V$ or $V' = \omega \overline{V}$. Here ω is some 4th root of unity and $\overline{V}$ is the complex conjugate of V. Typically there are 8 vectors in each equivalence class. We say that two matrices M and M' are equivalent if each column of M is equivalent to the corresponding column of M'. If we replace the matrix χ_j by an equivalent matrix χ_j' then we still recover X_j. Likewise, if we replace the matrix L_j by L_j' we still recover Λ_j.

Here is a criterion which will help us verify that the matrices we list give rise to double lattice PETs.

Lemma 5.1 *Suppose that χ and L are $n \times n$ matrices. Let $X = \chi(Q^{2n})$ and let Λ be the $\boldsymbol{Z}[i]$-span of the columns of L. Then X is a fundamental domain for Λ provided that $\chi^{-1}L$ is a triangular and has 4-th roots of unity along the diagonal.*

Proof: X is a fundamental domain for Λ if and only if $T(X)$ is a fundamental domain for $T(\Lambda)$. Here T is any invertible linear transformation of $\boldsymbol{R}^{2n} \approx \boldsymbol{C}^n$. In particular, this is true for $T = \chi^{-1}$. In other words, it suffices to consider the case when χ is the identity matrix and $X = Q^{2n}$. Replacing L by an equivalent matrix, we can assume that L has 1s along the diagonal. But Q^{2n} is indeed a fundamental domain for a lattice whose defining matrix is triangular and has 1s along the diagonal. ♠

We call $(\chi_1, \chi_2, L_1, L_2)$ a *special system* if $\chi_i^{-1} L_j$ is a triangular matrix with 4th roots along the diagonal for each pair $(i, j) \in \{1, 2\}$. The following corollary produces an n-parameter family of double lattice PETs from a special system.

[5]Not all double lattice PETs can be specified this way, but the ones here can be.

Corollary 5.2 *Let D be any nonsingular diagonal matrix. If $(\chi_1, \chi_2, L_1, L_2)$ is a special system, then $(\chi_1 D, \chi_2 D, L_1 D, L_2 D)$ is the data for a double lattice PET.*

Proof: We compute

$$(\chi_i D)^{-1}(L_j D) = D^{-1}(\chi_i^{-1} L_j) D = D^{-1} T D = T'. \tag{44}$$

Here T is the triangular matrix guaranteed by the hypotheses, and T' is the conjugate matrix. T' is also triangular, and has 4th roots of unity along the diagonals. So, for all indices i, j, the parallelotope $\chi_i D(Q^{2n})$ is a fundamental domain for the lattice defined by $L_j D$. ♠

Remark: We could produce an example of a special system by taking 4 upper triangular matrices (or lower triangular matrices). However, this would lead to a fairly trivial double lattice PET. The map would essentially be a parabolic linear transformation.

5.2 An Explicit Example

We discovered the construction in this section by working out the details of the compactifications described in the previous section. We will present the construction first and then identify it with what we did in the previous chapter. Our construction works for $n = 2, 6, 10, 14, ...$ We introduce matrices

$$\chi_\pm = \begin{bmatrix} \pm i & \pm i & \pm i & \pm i & \dots \\ +i & -1 & 0 & 0 & \dots \\ 0 & +i & -1 & 0 & \dots \\ 0 & 0 & +i & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{45}$$

$$L_\pm = \begin{bmatrix} \mp 1 & \pm i & \pm 1 & \mp i & \dots \\ +1 & -1 & 0 & 0 & \dots \\ 0 & +1 & -1 & 0 & \dots \\ 0 & 0 & +1 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{46}$$

The first row for $L_\pm$ repeats every 4 entries. Each entry in this first row is $(-i)$ times the preceding element. Just to be clear, the top left entry of L_+ is (-1). We set $\chi_1 = \chi_+$ and $\chi_2 = \chi_-$ and $L_1 = L_+$ and $L_2 = L_-$.

Now we will verify that $(\chi_1, \chi_2, L_1, L_2)$ is a special system. Letting J be the diagonal matrix with entries $(-1, +1, +1, +1, ...)$ we see that $\chi_2 = I\chi_1$ and $L_2 = IL_1$. We compute

$$\chi_2^{-1} L_1 = (I\chi_1)^{-1} L_2 = \chi_1^{-1} I^{-1} L_1 = \chi_1^{-1} I L_1 = \chi_1^{-1} L_2. \tag{47}$$

Similarly,

$$\chi_2^{-1} L_2 = \chi_1^{-1} L_1. \tag{48}$$

For this reason, we just have to check the conditions for $\chi_1^{-1} L_1$ and $\chi_1^{-1} L_2$. A routine calculation verifies that

$$L_1 = \chi_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1+i & 1 & 0 & 0 & 0 & 0 & \dots \\ -1-i & -1+i & 1 & 0 & 0 & 0 & \dots \\ +1-i & -1-i & -1+i & 1 & 0 & 0 & \dots \\ +1+i & +1-i & -1-i & -1+i & 1 & 0 & \dots \\ -1+i & +1+i & +1-i & -1-i & -1+i & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{49}$$

The matrix listed is such that entry in the lower triangle is $(-i)$ times the entry directly below it. Going down a column, the pattern has period 4. A similar calculation verifies that

$$L_2 = \chi_1 \begin{bmatrix} -i & -1+i & +1+i & +1-i & -1-i & -1+i & \dots \\ 0 & -i & -1+i & +1+i & +1-i & -1-i & \dots \\ 0 & 0 & -i & -1+i & +1+i & +1-i & \dots \\ 0 & 0 & 0 & -i & -1+i & +1+i & \dots \\ 0 & 0 & 0 & 0 & -i & -1+i & \dots \\ 0 & 0 & 0 & 0 & 0 & -i & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{50}$$

The matrix listed is such that each entry in the upper triangle is (i) times the entry immediately to its right. Going across a row, the pattern has period 4.

Thus, $(\chi_1, \chi_2, L_1, L_2)$ is a special system. if $S = \{s_1, ..., s_n\}$, we let D_S be the diagonal matrix whose diagonal entries are $s_1, ..., s_n$. We will see that the double lattice PET specified by $(\chi_1 D_S, \chi_2 D_S, L_1 D_S, L_2 D_S)$ is conjugate to the one produced by Theorem 1.7.

5.3 The Invariant Foliation

Choose some D_S. Let $X_j = \chi_j D_S(Q^{2n}))$ and let Λ_j be the lattice which is the $\boldsymbol{Z}[i]$ span of the columns of $L_j D_S$. Here we discuss some symmetries of the double lattice PET $(X_1, X_2, \Lambda_1, \Lambda_2)$. Let J be the involution mentioned in the previous section. By construction, the action of J swaps X_1 and X_2, and simultaneously swaps Λ_1 and Λ_2.

Considering J as a matrix acting on $\boldsymbol{R}^{2n}$, this map has a 2-dimensional (-1)-eigenspace and an $(2n-2)$-dimensional $(+1)$-eigenspace. This is just like the map L_g considered in connection with Theorem 1.7. The (-1)-eigenspace of J defines a complex line foliation $\mathcal{F}$ of X_1 (and X_2). The leaves of this foliation are parallel to $\boldsymbol{C} \times 0^{n-1}$.

Lemma 5.3 *The foliation $\mathcal{F}$ is invariant under the action of the double lattice PET.*

Proof: Let $\Pi = \boldsymbol{C} \times 0^{n-1}$. Since the matrices defining Λ_1 and Λ_2 agree below the first row, we see that the two sets $\Lambda_1(\Pi)$ and $\Lambda_2(\Pi)$ are identical. But this translates into the statement that the leaf of $\mathcal{F}$ through the origin is preserved by the double lattice PET. Finally, if one of the leaves is preserved, then all the leaves are preserved. ♠

Let $\mathcal{L}$ denote the leaf of $\mathcal{F}$ through the origin. The double lattice PET preserves $\mathcal{L}$ and induces an action on this real 2-dimensional space. Thus, even without knowing that we have simply recreated the compactifications from Theorem 1.7, we can see that the double lattice PETs here have associated planar actions. In the next section, we will identify these planar actions with the square turning maps from Theorem 1.7.

5.4 Connection to Theorem 1.7

We will work out the case $n = 6$ explicitly. This is a representative case. The cases $n = 10, 14, 18, \ldots$ follow the same pattern. We will treat the case $n = 2$ separately, and from a different point of view, in the next chapter. In our discussion, we flip back and forth between $\boldsymbol{C}^6$ and $\boldsymbol{R}^{12}$ using the identification $(z_1, \ldots, z_6) \leftrightarrow (x_1, y_1, \ldots, x_6, y_6)$. Our matrices are defined over $\boldsymbol{C}$, but $Q^{12} = [-1/2, 1/2]^{12}$ is defined over $\boldsymbol{R}$. Let

$$s_{ij} = \frac{s_i}{s_j}. \tag{51}$$

Referring to the construction in §4, the matrix L_g turns out to be

$$\begin{bmatrix} -i & (-1+i)s_{21} & (1+i)s_{31} & (1-i)s_{41} & (-1-i)s_{51} & (-1+i)s_{61} \\ (-1-i)s_{12} & i & (1+i)s_{32} & (1-i)s_{42} & (-1-i)s_{52} & (-1+i)s_{62} \\ (-1-i)s_{13} & (-1+i)s_{23} & 2+i & (1-i)s_{43} & (-1-i)s_{53} & (-1+i)s_{63} \\ (-1-i)s_{14} & (-1+i)s_{24} & (1+i)s_{34} & 2-i & (-1-i)s_{54} & (-1+i)s_{64} \\ (-1-i)s_{15} & (-1+i)s_{25} & (1+i)s_{35} & (1-i)s_{45} & -i & (-1+i)s_{65} \\ (-1-i)s_{16} & (-1+i)s_{26} & (1+i)s_{36} & (1-i)s_{46} & (-1-i)s_{56} & i \end{bmatrix}. \tag{52}$$

In general, the pattern is 4-periodic, except for a suitable shift in the indices of s_{ij}.

The matrices I_6 and L_g represent the two lattices Λ_1 and Λ_2. Here I_6 is the identity matrix. We seek a matrix M_1 which represents X_1 in the sense that $X_1 = M_1(Q^{12})$. Let $e_1, ..., e_6$ be the standard basis vectors. Looking at the proof of Lemma 4.8, we see that the first row of M_1^{-1} is e_1 and for $k > 1$ the kth row of M_1^{-1} is

$$L_1^t \circ ... \circ L_{k-1}^t(e_k). \tag{53}$$

Using this formula to compute M_1^{-1}, and then taking inverses, we see that M_1 is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (1-i)s_{12} & 1 & 0 & 0 & 0 & 0 \\ (1-i)s_{13} & (1-i)s_{23} & 1 & 0 & 0 & 0 \\ (1-i)s_{14} & (1-i)s_{24} & (1-i)s_{34} & 1 & 0 & 0 \\ (1-i)s_{15} & (1-i)s_{25} & (1-i)s_{35} & (1-i)s_{45} & 1 & 0 \\ (1-i)s_{16} & (1-i)s_{26} & (1-i)s_{36} & (1-i)s_{46} & (1-i)s_{56} & 1 \end{bmatrix} \tag{54}$$

Let $M_2 = L_g M$. The matrix data for our double lattice PET is given by (M_1, M_2, I_6, L_g).

Now we change coordinates. Let D_S and χ_1 be as in §5.2. Let $A = \chi_1 D_S$. We compute

$$AI_6 = \chi_1 D_S, \qquad AL_g = \chi_2 D_S, \qquad AM_1 = L_1 D_S \qquad AM_2 = L_2 D_S. \tag{55}$$

Thus, the double lattice PET $(A(X_1), A(X_2), A(\Lambda_1), A(\Lambda_2))$ is precisely the one constructed in §5.2. Note finally that

$$I := AL_g A^{-1} = J, \tag{56}$$

where J is the involution discussed in §5.3. This, A carries the invariant foliation for $(X_1, X_2, \Lambda_1, \Lambda_2)$ to the invariant foliation discussed in §5.3.

6 The Octagonal PETs

One should view this chapter as an elaboration of the case $n = 2$ from the previous chapter. Here we will take a different point of view. We are not sure if the cases $n = 6, 10, 14, ...$ can be treated in the same way we treat the case $n = 2$ here. This chapter mostly repeats material from [**S1**]. In this chapter, we use paramaters $\{1, s\}$ rather than $\{s_1, s_2\}$. This is our habit for the 2-grid systems.

Say that the *eigenlattice* of a parallelotope is the lattice generated by the vectors parallel to the sides of the parallelotope. Clearly a parallelotope is the fundamental domain for its eigenlattice. A *reflection* in a face of the parallelotope P is an order 2 linear isometry whose fixed set is a subspace parallel to a face of P. The face in question need not be a top-dimensional face. We prove the following elementary result in [**S1**, §3.2].

Lemma 6.1 (Reflection) *Let P be a parallelotope and Λ its eigenlattice. Let I be a reflection in a face of P. Then P is a fundamental domain for $I(\Lambda)$.*

6.1 The Real Case

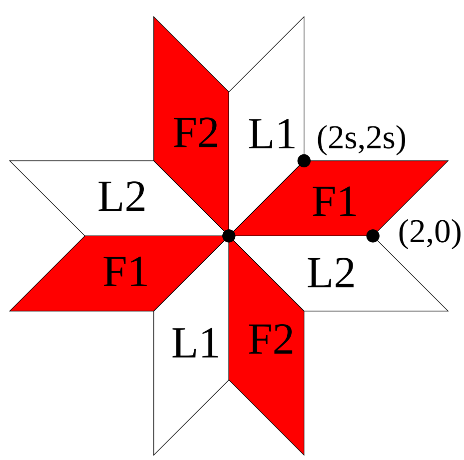

Figure 6.1: Scheme for the octagonal PET.

Figure 6.1 shows a scheme for a 2-dimensional double lattice PET. X_1 and X_2 are the origin-centered translates of F_1 and F_2 respectively. Λ_1 and Λ_2 are the eigenlattices respectively of L_1 and L_2.

Several applications of the Reflection Lemma show that $(X_1, X_2, \Lambda_1, \Lambda_2)$ really is the data for a double lattice PET. This double lattice PET depends on the parameter $s \in (0,1)$, which determines the shapes of our parallelograms. We call our system (X_s, f_s). Here is one of the many results we prove in [**S1**].

Theorem 6.2 *Let $s \in (0,1)$ be irrational. Then almost every point of (X_s, f_s) is periodic. The sequence of periods is unbounded, and the periodic islands are all semi-regular octagons or squares. For almost every $s \in (0,1)$, the system (X_s, f_s) has uncountably many aperiodic points.*

The last statement of the result does not automatically follow from the others because the map f_s is not everywhere defined. Below we will explain the proof for $s = \sqrt{2}/2$.

6.2 The Complex Case

In this section, we will define a 1-parameter family of 4-dimensional lattice PETs $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}})$ which, in a sense, are complexifications of the octagonal PETs. We remind the reader that a *real plane* in $\boldsymbol{C}^2$ is a 2-plane Π such that Π and $i\Pi$ are orthogonal. A *complex line* in $\boldsymbol{C}^2$ is a 2-plane Π such that $i\Pi$ and Π are parallel A *complex foliation* is a 2-dimensional foliation whose tangent planes are complex lines.

We let G denote the order 8 dihedral group generated by ρ_1 and ρ_2. Here

$$\rho_1(z_1, z_2) = (-z_1, z_2), \qquad \rho_2(z_1, z_2) = (z_2, z_1). \tag{57}$$

The invariant subspaces of ρ_1 and ρ_2 respectively are the complex lines given by $\{z_2 = 0\}$ and $\{z_1 = z_2\}$. The group G acts isometrically on $\boldsymbol{C}^2$. The restriction of G to $\boldsymbol{R}^2$ gives the group of symmetries of Figure 6.1.

Let $X_1^{\boldsymbol{C}}$ be the parallelogram centered at the origin and spanned by the vectors

$$(0,2), \qquad (0,2i), \qquad (s\zeta, s\zeta), \qquad (s\overline{\zeta}, s\overline{\zeta}); \qquad\qquad \zeta = 1+i. \tag{58}$$

The first two vectors lie in the complex line fixed by ρ_1 and the second two vectors lie in the complex line fixed by ρ_2. We let $X_2^{\boldsymbol{C}} = \rho_1 \circ \rho_2(X_1^{\boldsymbol{C}})$. We let $\Lambda_1^{\boldsymbol{C}}$ be the eigenlattice for $\rho_1(X_1^{\boldsymbol{C}})$. We could equally well describe $\Lambda_1^{\boldsymbol{C}}$ as the eigenlattice for $\rho_2(X_2^{\boldsymbol{C}})$. We let $\Lambda_2^{\boldsymbol{C}}$ be the eigenlattice for $\rho_2(X_1^{\boldsymbol{C}})$.

We could equally well describe $\Lambda_1^{\boldsymbol{C}}$ as the eigenlattice for $\rho_1(X_2^{\boldsymbol{C}})$. Note that $\Lambda_k^{\boldsymbol{C}} \cap \boldsymbol{R}^2 = \Lambda_k$. Several applications of the Reflection Lemma show that $(\boldsymbol{X}_1^{\boldsymbol{C}}, X_2^{\boldsymbol{C}}, \Lambda_1^{\boldsymbol{C}}, \Lambda_2^{\boldsymbol{C}})$ is a double lattice PET.

By construction, we have

$$X_k^{\boldsymbol{C}} \cap \boldsymbol{R}^2 = X_k, \qquad \Lambda_k^{\boldsymbol{C}} \cap \boldsymbol{R}^2 = \Lambda_k, \qquad k = 1, 2. \tag{59}$$

Here $(X_1, X_2, \Lambda_1, \Lambda_2)$ are the data for the octagonal PET at the parameter s. Hence, the octagonal PET (X_s, f_s) is an invariant "real slice" of $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}})$.

The complex octagonal PETs have quite a bit of symmetry. Let Γ be the order 8 dihedral group generated by the elements

$$\iota_1(z_1, z_2) = (\overline{z}_1, \overline{z}_2), \qquad \iota_2(z_1, z_2) = (iz_1, iz_2). \tag{60}$$

Each element of Γ preserves each parallelotope and each lattice defined in connection with the complex octagonal PETs. Hence, Γ acts as an order 8 group of symmetries of a complex octagonal PET.

There are 4 elements of Γ which act as real reflections – i.e., they pointwise fix real planes in $\boldsymbol{C}^2$. The planes $\Pi_0 = \boldsymbol{R}^2$ and $\Pi_2 = i\boldsymbol{R}^2$ are two of the fixed planes. The other two fixed planes are $\Pi_1 = \zeta\boldsymbol{R}^2$ and $\Pi_3 = \overline{\zeta}\boldsymbol{R}^2$. More simply,

$$\Pi_k = \zeta^k \boldsymbol{R}^2, \qquad k = 0, 1, 2, 3. \tag{61}$$

For instance, the map $\iota_2 \circ \iota_1$ fixes Π_1 pointwise. It turns out that

- $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}}) \cap \Pi_0$ is a copy of the octagonal PET (X_s, f_s).
- $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}}) \cap \Pi_2$ is copy of the octagonal PET (X_s, f_s).
- $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}}) \cap \Pi_1$ is a copy of the octagonal PET $(X_{s/2}, f_{s/2})$.
- $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}}) \cap \Pi_3$ is a copy of the octagonal PET $(X_{s/2}, f_{s/2})$.

Here, the word *copy* means *up to a similarity*. We sketch a proof of the last two assertions in [**S1**, §5]. We will not use them here, but we mention them to point out the beauty of the complex octagonal PETs.

Here is a concrete illustration. Figures 6.2 and 6.3 shows the picture of the periodic islands for the real octagonal PETs at the two parameters $s = \sqrt{2}/2$ and $s = \sqrt{2}/4$. Both pictures are self-similar and each figure appears as two of the slices mentioned above for parameter $s = \sqrt{2}/2$. Conjecturally, there is a

4-dimensional tiling by periodic islands, produced by the complex octagonal PET, which restricts to these tilings in the 4 slices. We will discuss this conjecture at the end of the chapter.

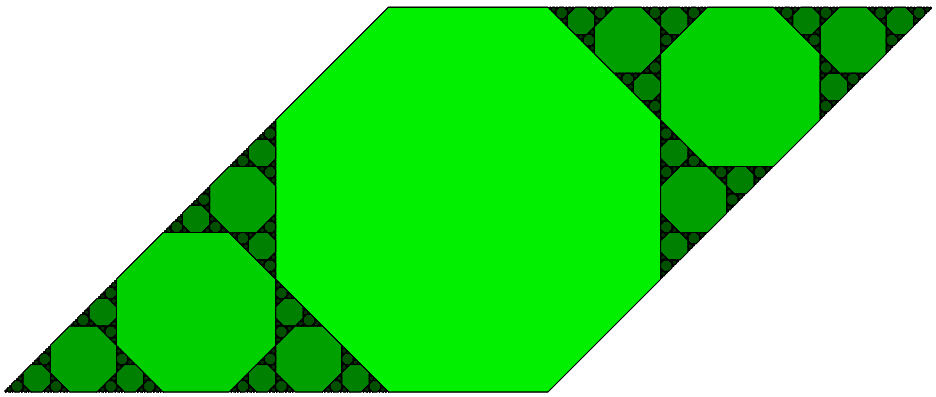

Figure 6.2: The tiling for $s = \sqrt{2}/2$

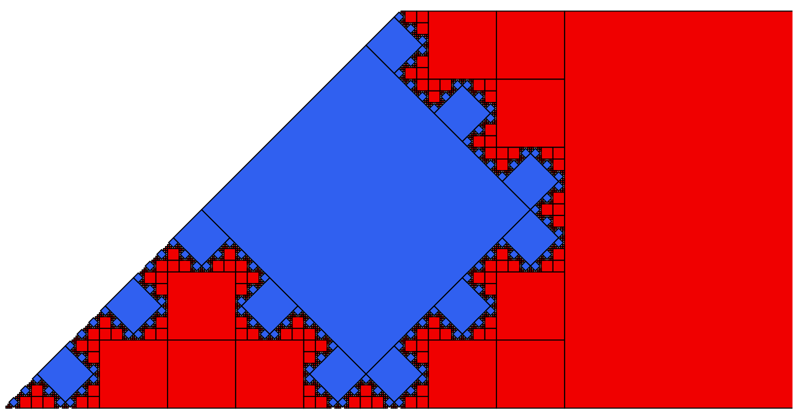

Figure 6.3: The tiling for $s = \sqrt{2}/4$.

These particular cases are very similar to other systems which arise in this kind of dynamics. Figure 6.2 is locally isometric to the tiling produced by the main example in [**AKT**], and also to the tiling produced by outer billiards on the regular octagon. Figure 6.3 is locally isometric to one of the tilings produced by the Truchet tile system in [**Hoo**].

Theorem 1.7 produces a double lattice PET for the word $g = (A_1 A_2)^2$ and for any irrational parameter s. In [**S1**, §4] we recognized this double lattice PET as the complex octagonal PET at parameter s. Combining Theorem 1.7 with this explicit calculation from [**S1**] we get the following corollary.

Corollary 6.3 *The complex octagonal PET* $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}})$ *has two invariant irrational orthogonal complex foliations,* F_1 *and* F_2*. Here*

1. F_1 *is spanned by* $(\zeta, \overline{\zeta})$ *and* $(\overline{\zeta}, -\zeta)$*.*

2. F_2 *is spanned by* $(\overline{\zeta}, \zeta)$ *and* $(\zeta, -\overline{\zeta})$*.*

For any $z \in \boldsymbol{C}$*, there is a leaf* $L = L(s, z)$ *of* F_1 *(or of* F_2*) such that the restriction of* $f_s^{\boldsymbol{C}}$ *to* L *is conjugate to the action of* $g_{s,z}$ *on* $\boldsymbol{C}$*. The map* $C \to L$ *is a piecewise isometry relative to the Euclidean metric on* $\boldsymbol{C}$ *and the path metric on* L*.*

Remark: Were we to glue together the opposite sides of $X^{\boldsymbol{C}}$, the map $C \to L$ would be an isometry.

Proof of Theorem 1.4: Let s be some irrational parameter and let $z \in \boldsymbol{C}$ be arbitrary. Let L be as in Corollary 6.3. Let $\{p_k\}$ be the unbounded sequence of periods associated to the real octagonal PET (X_s, f_s), as guaranteed by Theorem 6.2. For any k, there is some periodic island in (X_s, f_s) of period p_k. But (X_s, f_s) is an invariant slice of $(X_s^{\boldsymbol{C}}, f_s^{\boldsymbol{C}})$ and this latter system is a PET. Hence, the points of our 2-dimensional periodic island are contained in a 4-dimensional periodic island P of period p_k.

The piecewise isometry from $\boldsymbol{C}$ to L induces a locally Euclidean measure to L, relative to which $L \cap P$ has positive density in L. This density is exactly $\text{vol}(P)/\text{vol}(X)$. Hence, a positive density set of points in the invariant leaf L have period p_k. But then the same statement holds for points in $\boldsymbol{C}$, relative to the map $g_{z,s}$. Since $g_{s,z}$ has a positive density set of points of period p_k, it has infinitely many p_k-periodic islands. ♠

Proof of Theorem 1.5: We keep the same notation as in the proof of Theorem 1.4. This time, we choose s so that the octagonal PET (X_s, f_s) has uncountable many aperiodic points.

A routine calculation shows that the plane spanned by $(\overline{\zeta}, \zeta)$ and $(\zeta, -\overline{\zeta})$ is transverse to $\boldsymbol{R}^2$. Hence, the leaf L intersects $\boldsymbol{R}^2$ in a countable collection of points. The same goes for any leaf of F_1. Hence there uncountably many leaves of F_1 which contain aperiodic points of $X \subset \boldsymbol{R}^2$, the domain for the real octagonal PET. As in Lemma 3.7, this means that there are uncountably many choices of $z \in \boldsymbol{C}$ such that $L = L(s, z)$ contains an aperiodic point of the real octagonal PET.

Let $\omega : \boldsymbol{C} \to L$ be the conjugacy guaranteed by Corollary 6.3. Let $z \in \boldsymbol{C}$ be such that L contains an aperiodic point $x \in L \cap X$. The orbit of $O(x)$ of x is infinite, contained in X, and also contained in L. Let $x' = \omega^{-1}(x)$. By construction the orbit $O(x')$ under $g_{s,z}$ is infinite. Moreover ω carries $O(x')$ to $O(x)$. We want to see that $O(x')$ is unbounded.

If B is any bounded subset of $\boldsymbol{C}$, then $\omega(B)$ intersects X in only finitely many points. This follows from the fact that ω is an isometry when $X^{\boldsymbol{C}}$ is interpreted as a torus, and the image $L = \omega(\boldsymbol{C})$ is transverse to X. Since $\omega(B)$ can only contain finitely many points of $O(x)$, the bounded set B cannot contain all of $O(x')$. Hence $O(x')$ is unbounded. ♠

6.3 The Renormalization Scheme

Theorem 6.2 derives from a *renormalization scheme* we found for the (real) octagonal PETs. Let (X_s, f_s) be the octagonal PET at parameter s. Given $Y \subset X$ (and suppressing the parameter) let $f|Y$ denote the first return map of f to Y, assuming that this map is well-defined.

Define the map $R : (0,1) \to [0,1)$ by the formula

- $R(s) = 1 - s$ if $s \in [1/2, 1)$.
- $R(s) = 1/(2s) - \text{floor}(1/(2s))$ if $s \in (0, 1/2)$.

For all but countably many choices of s, we have $t = R(s) > 0$. In all these cases, we prove the following result.

Theorem 6.4 *There are clean convex polygons $Y_t \subset X_t$ and $Z_s \subset X_s$ and a similarity $\phi_s : Y_t \to Z_s$ such that*

- *ϕ_s conjugates $f_t|Y_t$ to $f_s^{-1}|Z_s$.*
- *Every nontrivial orbit of f_t intersects Y_t and every nontrivial orbit of f_s, escept possibly for certain orbits of period 2, intersects Z_s.*

The map ϕ_s is an isometry when $s > 1/2$ and a contraction when $s < 1/2$.

A polygon is *clean* if its boundary does not intersect any of the open periodic islands. We describe Y and Z explicitly in [**S1**].

For any given parameters s and $t = R(s)$, the proof of Theorem 6.4 is a fairly easy and finite calculation. I will explain this for the parameter

$s = \sqrt{2}/2$. In this case, Theorem 6.2 follows almost immediately from the self-similar nature of Figure 6.2.

Figure 6.4 shows the partitions defining f. The map f carries each polygon in the left partition by translation to a copy in the right partition.

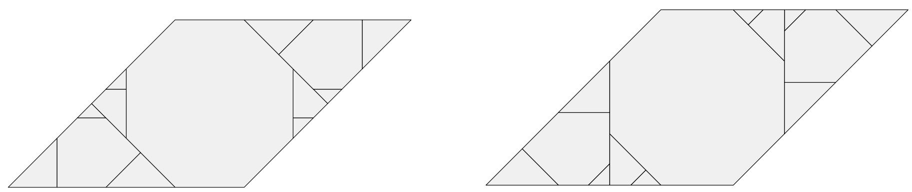

Figure 6.4: The partitions defining (X_s, f_s) for $s = \sqrt{2}/2$.

Inspecting the figure, we see that the central octagon O_1 is the fixed island of f and that there are two smaller octagonal islands O_{21} and O_{22} of period 2, as shown in Figure 6.5.

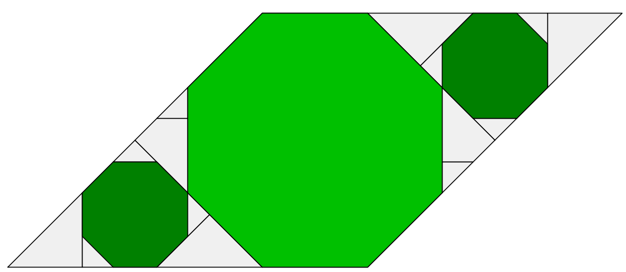

Figure 6.5: The period islands O_1 and O_{21} and O_{22}.

There are 8 sets of interest to us. $X - O_1$ is a union of two kites K_{11} and K_{12} and $X - O_1 - O_2$ is a union of 6 smaller kites $K_{21}, .., K_{26}$. There are two similarities carrying K_{1j} to K_{2k} for $j \in \{1, 2\}$ and $k \in \{1, ..., 6\}$. One can check with a finite calculation that each of these 12 maps conjugates $f|K_{1j}$ either to $f|K_{2k}$ or $f^{-1}|K_{2k}$. This property implies that the tiling by periodic islands is invariant under the action of these 12 maps. The rest of Figure 6.2 is then determined by this structure.

We explain one of these calculations. The rest are similar, and indeed follow from symmetry. Define.

$$K_j = K_{j1} \cup \rho_1(K_{j1}), \qquad j = 1, 2. \tag{62}$$

Here ρ_1 is reflection in the origin. Let ϕ be the piecewise orientation-reversing similarity which carries the left (respectively right) half of K_1 to the left (respectively right) half of K_2. If one can verify that

$$\phi^{-1} \circ (f^{-1}|K_2) \circ \phi = f|K_1 \tag{63}$$

then the same relation holds with K_{j1} in place of K_j.

Let Π_+ denote the forward partition for f, shown in Figure 6.4. Say that a finite sequence of points $p_1, ..., p_k$ is *feasible* if there are polygons $P_1, ..., P_k$ in the partition Π_+ such that $p_j \in \overline{P}_j$ for all j and $f_j(p_j) = p_{j+1}$ for $j = 1, ..., k-1$. Here f_j is the extension of $f|P_j$ to the closure $\overline{P}_j$. A finite portion of a genuine orbit is feasible, and so are the limits of such things. However, a feasible sequence may not be a portion of a well-defined orbit. We call another feasible sequence $q_1, ..., q_k$ *compatible* with $p_1, ..., p_k$ if $q_j \in \overline{P}_j$ for all j. That is, both sequences visit the same sequence of partition polygons.

Lemma 6.5 *Suppose that Q is a n-gon, with vertices $p_{11}, ..., p_{n1}$. If there exist n mutually compatible sequences $p_{j1}, ..., p_{jk}$ for $j = 1, ..., n$, then f^k is well defined on all points on the interior of Q.*

Proof: Let $P_1, ..., P_k$ be the sequence of polygons of Π_+ visited by our sequences. Let q_1 be a point in the interior of Q. q_1 lies in the interior of P_1 by convexity. Hence f is defined on q_1 and $q_2 = f(q_1) \in P_2$. And so on. ♠

We will use Lemma 6.5 to verify the fact that K_2 is an invariant set for f^{-3} and that ϕ conjugates the action of f on K_1 to the action of f^{-3} on K_2. Let g be the map on the left hand side of Equation 63. Using the definedness criterion, with respect to the inverse map f^{-1} and the inverse partition Π_-, we check that f^{-3} is well defined on every polygon of the form $\phi(\Pi_+|K)$. But this means that both f and g are well defined (and hence translations) on the interior of each of the 12 polygons of $\Pi_+ - O_1$. But then it suffices to check Equation 63 on 12 points, one per polygon. We omit the details of these few calculations.

7 The Weyl Pseudogroup

In this chapter I will describe some work in progress, which should lead to a proof of Conjecture 1.1. Details will appear in a forthcoming paper, if I can work it all out.

One of the main things we would like to show is that, for every parameter $s \in (0,1)$, almost every point of the complex octagonal PET (X_s, f_s) is periodic. This, X_s should have an a.e. defined tiling by periodic islands. Moreover, this tiling should restrict to copies of the already-established a.e. periodic tiling on the 4-real slices discussed above. However, this tiling has some "glitches". When restricted to the two irrational foliations from Corollary 6.3, it produces Pictures like Figure 1.1. These glitches prevent the existence of a renormalization result which extends the one in the real case.

In studying the (imperfect) symmetries of the periodic island tiling, I hit upon a new kind of dynamical system which enhances the complex octagonal PETs, has a renormalization scheme, and exhibits the same kind of beautiful symmetry (without glitches) that one sees in the real case. In this section I will sketch that system and state its properties. So far I don't have any proofs of the assertions, but I do have overwhelming experimental evidence.

7.1 Pseudogroup Actions

Let $X \subset \boldsymbol{R}^d$ be a compact domain and let $g_1, ..., g_n$ be some finite list of isometries. We insist that this list is symmetric with respect to inverses. In other words, an isometry appears on the list if and only if the inverse isometry appears on the list. We do not require that $g_k(X) = X$.

For each k, we define

$$X_k = g^{-1}(g_k(X) \cap X). \tag{64}$$

By definition, $X_k \subset X$ is the maximal subset of X such that $g_k(X_k) \subset X$. We call a subset $Y \subset X$ *stable* if it has for following property. If $p \in Y \cap X_k$ then $g_k(p) \in Y$ as well. In other words, the pseudogroup action maps Y into itself. We call Y a *pseudogroup orbit* if Y is stable and no proper subset of Y is stable. In case all the isometries preserve X, a pseudogroup orbit corresponds with an orbit of the group generated by the isometries.

It is tricky to say what we mean by a periodic island because a pseudogroup action is not a dynamical system in the traditional sense. Here is a provisional definition which is adapted to our examples below. For each

p, let O_p^* denote the pseudogroup orbit containing p. We say that a *periodic island* is a convex polytope $S \subset X$ such that the union

$$\bigcup_{P \in S} O_p^*$$

is a finite union of pairwise disjoint isometric copies of S. In our examples below, each pseudogroup orbit intersects a periodic island in either 24 or 192 points.

7.2 Examples

Let $s \in (0,1)$. Let $X = X_1^{\boldsymbol{C}}$ denote the domain defined above at the parameter s. Consider the matrix

$$A = \frac{s}{2} \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 \end{bmatrix} \tag{65}$$

This matrix is a multiple of a Hadamard matrix. The rows and columns are orthogonal and all have the same length, namely s.

Let Π denote the tiling of $\boldsymbol{R}^4$ by unit cubes, such that the origin is a vertex of one of the cubes. Let Γ_1 denote the infinite list of elements generated by reflections in the hyperplanes of Π. By construction, Γ_1 is an infinite Bieberbach group. Let

$$\Gamma_2 = A \circ \Gamma_1 \circ A^{-1}. \tag{66}$$

Then Γ_2 is the group generated by reflections in the hyperplanes of the smaller and rotated grid $A(\Pi)$. No matter how s is chosen, there are only finitely many elements $g \in \Gamma_1 \cup \Gamma_2$ such that $g(X) \cap X \neq \emptyset$. To be clear, the group generated by $\Gamma_1 \cup \Gamma_2$ is a dense group, and there are infinitely many elements of this group within every neighborhood of the identity. However, we are only allowed to take elements in $\Gamma_1 \cup \Gamma_2$.

We call $g \in \Gamma_1 \cup \Gamma_2$ *good* if either $g(X) \cap X \neq \emptyset$ or $g^{-1}(X) \cap X \neq \emptyset$. We call the pseudogroup generated by the good elements the *Weyl pseudogroup*.

7.3 Properties

Let O_p and O_p^* respectively denote the complex octagonal PET orbit and the Weyl pseudogroup orbit of $p \in X$. Let $|O_p|$ and $|O_p^*|$ be the cardinalities of O_p and O_p^*. I observed the following experimentally.

1. For almost all p, orbit O_p^* is finite and $|O_p^*|$ is a multiple of 192.
2. For every choice of p we have $O_p \subset O_p^*$. In general $192|O_p| < |O_p^*|$.
3. $O_p^* \cap \Pi_j = O_p \cap \Pi_j$ for all $p \in X \cap \Pi_j$ and all $j = 1, 2, 3, 4$.

Here $\Pi_1, ..., \Pi_4$ are the real octagonal PET slices discussed above. What these properties say is that the Weyl pseudogroup action is an extension of the real octagonal PET which is compatible (but different from) the complex octagonal PET.

There is a certain finite reflection group Γ of order 192 which plays a special role in our discussion. Γ is the group of reflections in the faces of the 24-cell, the 4-dimensional platonic solid whose vertices are the integer vectors of length $\sqrt{2}$ consisting of permutations of the vectors $(\pm 1, \pm 1, 0, 0)$. The group Γ is sometimes called the D_4 Weyl group.

For each irrational parameter s it seems that all the periodic islands have Γ-symmetry, and that almost every point is contained in a periodic island. (I can draw the tiles for small parameters on the computer.) Call this tiling $\mathcal{T}_s$. The following seems to be true:

1. Each pseudogroup orbit intersects each island in $\mathcal{T}_s$ in 192 points. These points make a single orbit under the symmetry group conjugate to Γ which acts on the tile.
2. $\mathcal{T}_s$ intersects each real slice Π_j in the periodic island tiling of the corresponding real octagonal PET.
3. $\mathcal{T}_s$ refines the conjectured periodic island tiling associated to the complex octagonal pet at parameter s, in the sense that each octapet periodic island is a union of tiles of $\mathcal{T}_s$.
4. $\mathcal{T}_s$ intersects a certain square in the irrational foliation from Corollary 6.3 in a rectangle pattern which reflects the continued fraction expansion of s and has no "glitches" like Figure 1.1. In other words, the Weyl pseudogroup action "corrects" the picture made by the alternating grid system.

7.4 Renormalization Scheme

All the structure above seems to derive from what appears to be a renormalization scheme for the pseudogroup tiling: Let

$$t = 1/s - \text{floor}(1/s).$$

the image of s under the Gauss map. It seems that there are subsets $Y_t \subset X_t$ and $Z_s \subset X_s$ and a similarity $\phi_s : Y_t \to Z_s$ such that

$$\phi_s(\mathcal{T}_t \cap Y_t) = \mathcal{T}_s \cap Z_s. \tag{67}$$

The map ϕ_s is a contraction for all s, and the linear part of ϕ_s is a multiple of the Hadamard matrix given above. The set Y_t is just the intersection of X_t with a particular linear halfspace. (My computer program draws the sets and map exactly.)

On the level of pseudogroup orbits, we seem to have

$$\phi_s(O_p^* \cap Y_t) \subset O_{\phi(p)}^* \cap Z_s. \tag{68}$$

However, we do not always have equality in this last equation. I still do not know exactly what is going on, and I don't have a proof yet.

The renormalization scheme mentioned here should lead to proof of Conjecture 1.1 just as Theorem 6.4 leads to all the structure of the real octagonal PETs derived in [**S1**]. However, I am far from working out the details of this.

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