The Octagonal PET II: The Topology of the Limit Sets

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Abstract

This paper is a sequel to [**S0**]. In [**S0**], we studied a 1-parameter family of polygon exchange transformations. We showed that this family is completely renormalizable. In this paper, we use the renormalization scheme in [**S0**] to give a complete classification of topological types of the limit sets which arise in these PETs.

1 Introduction

1.1 Background

A *polytope exchange transformation* (or PET) is defined by a polytope X which has been partitioned in two ways into smaller polytopes:

$$X = \bigcup_{i=1}^{m} A_i = \bigcup_{i=1}^{m} B_i.$$

For each i there is some vector V_i such that $B_i = A_i + V_i$. That is, some translation carries A_i to B_i. One then defines a map $f : X \to X$ by the formula $f(x) = x + V_i$ for all $x \in \text{int}(A_i)$. It is understood that f is not defined for points in the boundaries of the small polytopes. The inverse map is defined by $f^{-1}(y) = y - V_i$ for all $y \in \text{int}(B_i)$.

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The simplest examples of PETs are 1-dimensional systems, known as *interval exchange transformations* (or IETs). These systems have been extensively studied in the past 30 years, and there are close connections between IETs and other areas of mathematics such as Teichmuller theory. See, for instance, [**Y**] and [**Z**] and the many references mentioned therein.

Some examples of polygon exchange maps have been studied in [**AG**], [**AKT**], [**H**], [**Hoo**], [**LKV**], [**Low**], [**S2**], [**S3**], and [**T**]. Some definitive theoretical work concerning the (zero) entropy of such systems is done in [**GH1**], [**GH2**], and [**B**].

The *Rauzy renormalization* [**R**] gives a satisfying renormalization theory for the family of IETs all having the same number of intervals in the partition. Often, renormalization phenomena are found in individual PETs, and in the papers [**S0**], [**S1**] and [**Hoo**], there were constructed families of PETs which had a renormalization theory at least vaguely similar to the Rauzy renormalization.

In [**S0**], we introduced a family of PETs in every even dimension. In dimension $2n$, the objects are indexed by $GL_n(\boldsymbol{R})$. In the two-dimensional setting, there is a 1-parameter family. We studied this family in detail, and in particular worked out a renormalization scheme for the family. In this paper, we use the renormalization scheme to work out the topology of the associated limit sets of the 2-dimensional examples.

In [**S0**] we give a somewhat fuller discussion of the background and context for our examples.

1.2 Construction of the PET

Here we recall the basic construction given in [**S0**], at least in the 2-dimensional case. Our construction depends on a parameter $s \in (0, \infty)$. The 8 parallelograms in Figure 1.1 are the orbit of a single parallelogram P under a dihedral group of order 8. Two of the sides of P are determined by the vectors $(2,0)$ and $(2s, 2s)$. For $j = 1, 2$, let F_j denote the parallelogram centered at the origin and translation equivalent to the ones in the picture labeled F_j. Let L_j denote the lattice generated by the sides of the parallelograms labeled L_j. (Either one generates the same lattice.)

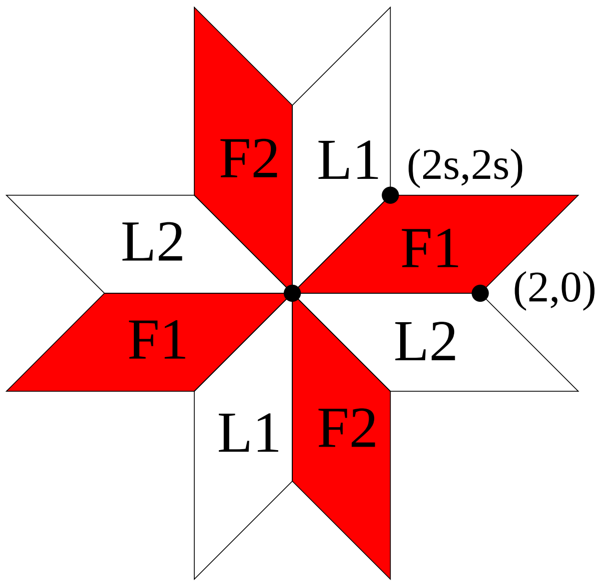

Figure 1.1: The scheme for the PET.

It turns out that F_i is a fundamental domain for L_j, for all $i,j \in \{1,2\}$. Given $p \in F_j$ we let

$$f'(p) = p + V_p \in F_{3-j}, \qquad V_p \in L_{3-j}. \tag{1}$$

The choice of V_p is almost always unique, on account of F_{3-j} being a fundamental domain for L_{3-j}. When the choice is not unique, we leave f' undefined. When $p \in F_1 \cap F_2$ we have $V_p = 0 \in L_1 \cap L_2$. We define $f = (f')^2$, which preserves both F_1 and F_2, and we set $X = F_1$. Our system is $f : X \to X$, which we denote by (X, f).

1.3 Prior Results

The results in [**S0**] (and in this paper) ultimately follow from a renormalization scheme associated to our family. Define the *renormalization map* $R : (0,1) \to [0,1)$ as follows.

- $R(x) = 1/(2x) - \text{floor}(1/(2x))$ if $x < 1/2$.
- $R(x) = 1 - x$ if $x > 1/2$.

This map will appear in many of our results. We explain its direct connection to the PETs in §2.

A *periodic tile* for (X, f) as a maximal convex polygon on which f and its iterates are completely defined and periodic. We call the union Δ of the periodic tiles the *tiling*. In [**S0**] we proved, among other things, the following results about the tiling.

Theorem 1.1 *When s is irrational, Δ_s is a full measure set consisting entirely of squares and semi-regular octagons.*

- *Δ_s consists entirely of squares iff $R^n(s) < 1/2$ for all n.*
- *Δ_s has finitely many octagons iff $R^n(s) > 1/2$ for finitely many n.*
- *Δ_s has infinitely many octagons iff $R^n(s) > 1/2$ for infinitely many n.*

The fact that the union of tiles in Δ_s has full measure allows us to define the limit set [1] $\widehat{\Lambda}_s$ to be the set of points p such that every neighborhood of p intersects infinitely many tiles of Δ.

In [**S0**] we proved, among other things, the following results about the limit set.

Theorem 1.2 *Let $s \in \boldsymbol{R}$ be irrational. The projection of $\widehat{\Lambda}_s$ onto any line parallel to an 8th root of unity contains a line segment. Hence $\widehat{\Lambda}_s$ has Hausdorff dimension at least 1. Moreover, $\widehat{\Lambda}_s$ is not contained in a finite union of lines.*

We also proved that our 1-parameter family had what one might call hidden hyperbolic symmetry.

Theorem 1.3 *The Hausdorff dimension of $\widehat{\Lambda}_s$, as a function of the parameter, is invariant under the action of the $(2, 4, \infty)$ hyperbolic reflection triangle group acting on the parameter space by linear fractional transformations.*

The results in [**S0**] are somewhat more detailed than the ones listed above, and the reader should refer to [**S0**] for the more detailed versions of the results above.

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[1] For a general polygon exchange map, the limit set is probably best defined as the set of weakly aperiodic points. We call a point $p \in X$ *weakly aperiodic* if there is a sequence $\{q_n\}$ converging to p with the following property. The first iterates of f are defined on q_n and the points $f^k(q_n)$ for $k = 1, ..., n$ are distinct.

1.4 Topology of the Limit Sets

Here is the main result of this paper.

Theorem 1.4 (Main) *Let $s \in (0,1)$ be irrational.*

1. *$\widehat{\Lambda}_s$ is a disjoint union of two arcs if and only if $R^n(s) < 1/2$ for all n.*
2. *$\widehat{\Lambda}_s$ is a finite forest* [2] *if and only if $R^n(s) > 1/2$ for finitely many n.*
3. *$\widehat{\Lambda}_s$ is a Cantor set if and only if $R^n(s) > 1/2$ infinitely often.*

Combining the Main Theorem with Theorem 1.1, we get the following corollary.

Corollary 1.5 *Let $s \in (0,1)$ be irrational.*

1. *$\widehat{\Lambda}_s$ is a disjoint union of two arcs if and only if Δ_s contains only squares.*
2. *$\widehat{\Lambda}_s$ is a finite forest if and only if Δ_s contains finitely many octagons.*
3. *$\widehat{\Lambda}_s$ is a Cantor set if and only if Δ_s contains infinitely many octagons.*

Figures 1.2 shows Statement 1 of the Main Theorem in action.

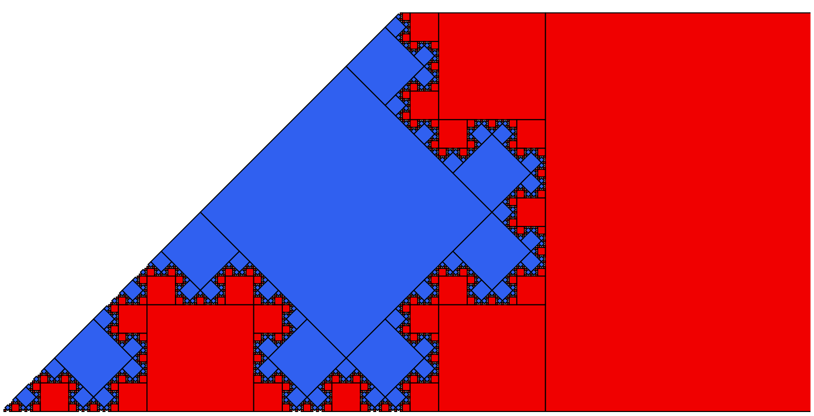

Figure 1.2: The left half of Δ_s for $s = \sqrt{3}/2 - 1/2$.

[2] A *finite forest* is a finite disjoint union of finite trees

Figure 1.3 shows Statement 2 of the Main Theorem in action. The parameter is such that its even expansion is $3, 1, 3, 1, 2, 2, \ldots$. A k in the nth position indicates that $R^n(s) \in (1/(k+1), 1/k)$.

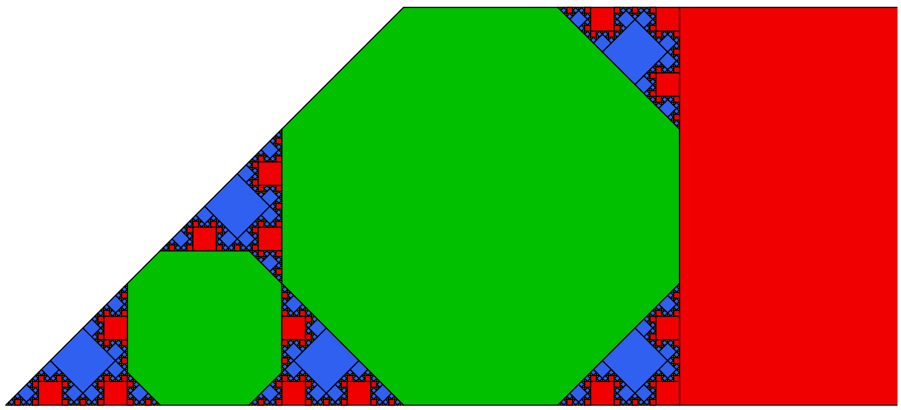

Figure 1.3: The left half of Δ_s for $s = (3, 1, 3, 1, 2, 2, 2 \ldots)$.

Figure 1.4 shows Statement 3 of the Main Theorem in

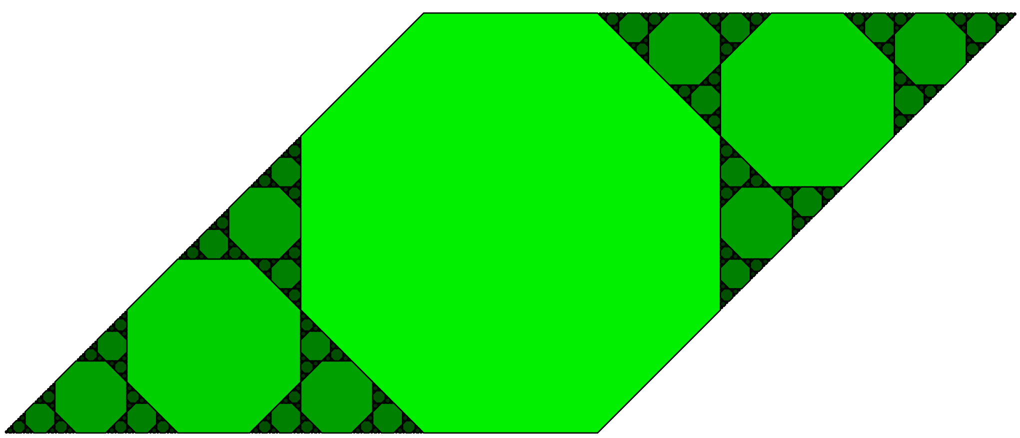

Figure 1.3: The left half of Δ_s for $s = \sqrt{2}/2$.

In all these figures, the right half is a rotated image of the left half. Intuitively, the squares tend to line up in simply connected chunks whose complementary regions are curves and the octagons both split and splice the curves.

This paper is organized as follows. In §2 we will recall the renormalization scheme from [**S0**], as well as some symmetry results. Following the summary given in §2, the rest of this paper is self-contained. In §3-4 we prove some geometric and combinatorial results about how the tiles of Δ_s fill X_s and how $\widehat{\Lambda}_s$ intersects various lines of symmetry. In §5-7 we prove the Main Theorem, one statement per chapter.

We would like to say a word about the nature of our proofs. For the sake of giving a readable exposition, there are a number of routine geometric calculations, concerning polygonal regions in the plane, which we omit. Such calculations are all exercises in plane geometry. Rather than write out these calculations, we illustrate them with extensive pictures from our java program OctaPET. It is certainly best to read this paper while using OctaPET; this is how we wrote the paper.

A common mistake made by beginning students is to try to prove a general statement by just considering one example. We do not mean to make this mistake here, even though superficially some of our proofs look like this. In a written paper dealing with a 1-parameter family of systems, we cannot illustrate the picture for every parameter. The pictures we do show are typical for the given parameter interval, and the written arguments we give only make statements which hold for all the relevant parameters.

I would like to thank Nicolas Bedaride, Pat Hooper, Injee Jeong, John Smillie, and Sergei Tabachnikov for interesting conversations about topics related to this work. Some of this work was carried out at ICERM in Summer 2012, and some was carried out during my sabbatical in 2012-13. This sabbatical was funded from many sources. I would like to thank the National Science Foundation, All Souls College, Oxford, the Oxford Maths Institute, the Simons Foundation, the Leverhulme Trust, and Brown University for their support during this time period.

2 Preliminaries

2.1 Closedness

We start with an essentially obvious result, which we state for the record.

Lemma 2.1 $\widehat{\Lambda}_s$ *is closed.*

Proof: Let $\{p_n\}$ be a sequence of points in $\widehat{\Lambda}_s$ converging to some q. We want to show that $q \in \widehat{\Lambda}_s$. We need to show that every open neighborhood U of q contains infinitely many tiles of Δ_s. For some n we have $p_n \in U$. But then some neighborhood V of p_n lies in U. But V contains infinitely many tiles of Δ_s. Hence, so does U. ♠

2.2 The Even Expansion

We call an irrational number $s \in (0,1)$ is *oddly even* if it has continued fraction expansion $[0, a_1, a_2, a_3, ...]$ with a_k even for all odd k. In [**S0**] we showed that s is oddly even if and only if $R^n(s) < 1/2$ for all n. We call the rational s *oddly even* if the (finite) orbit $\{R^n(s)\}$ avoids $(1/2, 1)$. In [**S0**], we proved the following easy-to-believe lemma.

Lemma 2.2 *Let s be an oddly even irrational. There is a sequence $\{r_n\}$ of oddly even rational numbers which converges to s.*

When s is irrational, we define the *even expansion* of s to be $n_0, n_1, n_2, ...$ where

$$\frac{1}{n_k + 1} < R^k(s) < \frac{1}{n_k}. \tag{2}$$

Since R maps each interval $(1/2n, 1/(2n-1)]$ onto $[1/2, 1)$, an irrational $s \in (0,1)$ is oddly even if and only if its even expansion consists entirely of even numbers.

We can define the even expansion of $s \in (0,1)$ when s is rational. The last nonzero term in the R-orbit of s is $1/2q$ for some $q = 1, 2, 3...$ and by convention we take the corresponding term in the even expansion to be $1/2q$. The preceding terms lie in the interiors of the intervals in Equation 2. Wnen $r_n \to s$ and s is irrational, even expansions of r_n converges to the even expansion of s.

2.3 Rational Limits

When we speak of the convergence of compact sets, we mean to use the Hausdorff topology. A sequence of compact sets $\{P_n\}$ converges to a compact set P_∞ if, for every $\epsilon > 0$ there is some N such that $n > N$ implies that the Hausdorff distance from P_n to P_∞ is less than ϵ. The Hausdorff distance between compact sets S_1 and S_2 is the infimal d such that S_j is contained in the d-neighborhood of S_{3-j} for $j = 1, 2$.

Let $\{s_n\}$ be a sequence of rationals converging to an irrational parameter $s \in (0, 1)$. Let Δ_n be the perodic tiling associated to s_n and let Δ_∞ be the periodic tiling associated to s.

Lemma 2.3 (Approximation) *Let P_∞ be a tile of Δ_∞. Then for all large n, there is a tile P_n of Δ_n such that $\{P_n\}$ converges to P_∞.*

Proof: We proved this in [**S0**, §3,3]. Here is a sketch. When s is irrational, we show that the periodic points are stable under perturbation. Hence, there is some sequence $\{P_n\}$ such that $\lim P_n$ contains an open subset of P_∞. We also show that the set of periodic points in Δ_t with the same combinatorics (essentially the list of vectors we use when defining the map f_n and its iterates, written in a parameter-independent basis) is a single tile of Δ_t, as long as $t \neq 1/n$ for $n = 1, 2, 3....$ That is, different tiles have different associated dynamical combinatorics. This uniqueness forces $\lim P_n = P_\infty$. ♠

Lemma 2.4 *Let P_n be a tile of Δ_n and $\{P_n\}$ converges to a polygon P_∞. Then P_∞ is a tile of Δ_∞.*

Proof: We give the argument in [**S0**, §6.2]. Here is a sketch. There is a uniform lower bound on the size of P_n and a uniform upper bound on the area of X_n. This puts a uniform upper bound on the period of P_n. Passing to a subsequence, we can assume that the combinatorics of the orbit is independent of n. By continuity, any point in P_∞ is a periodic point for f_n when n is sufficiently large. This shows that there is a periodic tile P'_∞ of Δ_∞ which contains P_∞. Applying the Convergence Lemma, there is some sequence $\{P'_n\}$ such that $P'_n \to P'_\infty$. Eventually P'_n and P_n overlap, and so we must have $P_n = P'_n$. But then $P'_\infty = P_\infty$. ♠

We abbreviate the results in this section by saying that the tilings $\{\Delta_n\}$ converge to the tiling Δ_∞.

2.4 Renormalization

In this section we explain the significance of the renormalization map R which appears in our results.

We use the notation from §1. So, $F_1 = X$ is the domain of our system and F_2 is the image of F_1 rotated by $\pi/2$. The map f is trivial on the intersection $F_1 \cap F_2$. We call this intersection the *trivial tile*.

Suppressing the parameter s, we define

$$Y = F_1 - F_2 = X - F_2 \subset X. \tag{3}$$

Y is the portion of X outside the trivial tile.

We call a subset $S \subset X$ *clean* if ∂S does not intersect the interior of any tile of Δ. Here is the main result from [**S0**].

Theorem 2.5 *Suppose $s \in (0,1)$ and $t = R(s) \in (0,1)$. There is a clean set $Z_s \subset X_s$ and a similarity $\phi_s : Y_t \to Z_s$ such that $\phi_s(\Delta_t \cap Y_t) = \Delta_s \cap Z_s$.*

1. *ϕ_s commutes with reflection in the origin and maps the acute vertices of X_t to the acute vertices of X_s.*

2. *When $s < 1/2$, the restriction of ϕ_s to each component of Y_t is an orientation reversing similarity, with scale factor $s\sqrt{2}$.*

3. *When $s < 1/2$, either half of ϕ_s extends to the trivial tile of Δ_t and maps it to a tile in Δ_s.*

4. *When $s < 1/2$, the only nontrivial orbits which miss Z_s are contained in the ϕ_s-images of the trivial tile of Δ_t. These orbits have period 2.*

5. *When $s > 1/2$ the restriction of ϕ_s to each component of Y_s is a translation.*

6. *When $s > 1/2$, all nontrivial orbits intersect Z_s.*

Figures 2.1 and 2.2 illustrate the result for $s < 1/2$. Figures 2.3 and 2.4 show the Main Theorem in action for $s > 1/2$.

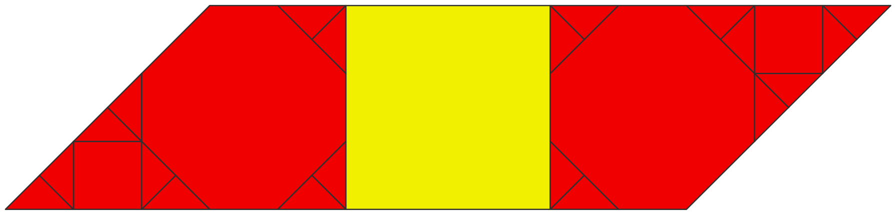

Figure 2.1: Y_t in red for $t = 3/10 = R(5/13)$.

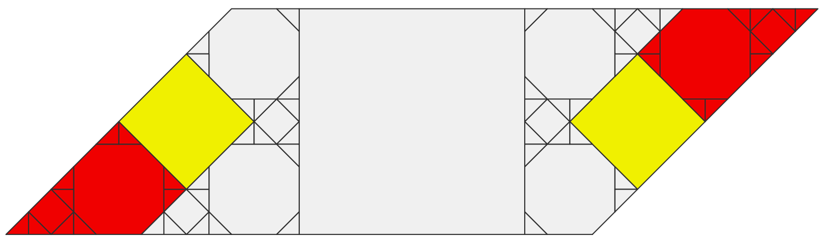

Figure 2.2: Z_s in red $s = 5/13$.

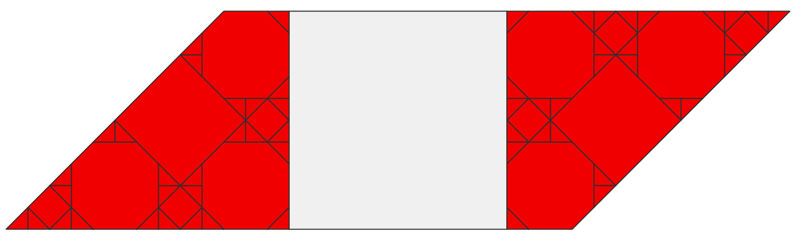

Figure 2.3: Y_t in red for $t = R(8/13) = 5/13$.

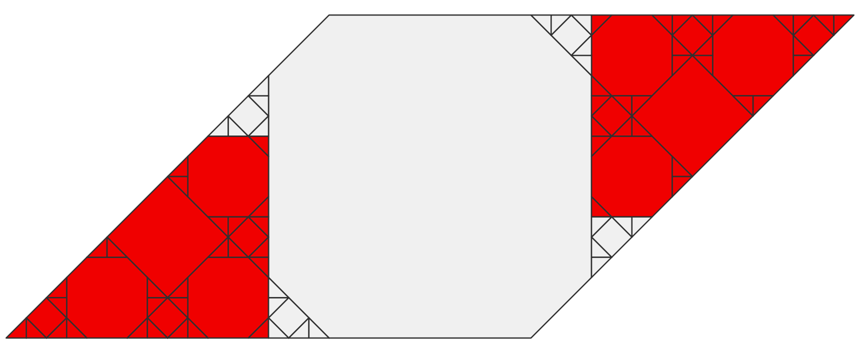

Figure 2.4: Z_s in red for $s = 8/13$.

2.5 Elementary Symmetry

In [**S0**] we established a number of symmetries of the system. The ones listed in this section have easy proofs, though we do not give them here.

Rotation Symmetry: Δ_s is invariant under reflection in the origin:

$$\iota(x,y) = (-x,-y), \tag{4}$$

In view of this symmetry, we will often draw only the left half of the picture.

Inversion Symmetry: Let $t = 1/(2s)$. There is a similarity carrying Δ_s to Δ_t. The similarity is orientation reversing, fixes the origin, and interchanges lines of slope 0 with lines of slope 1. We called this fact the Inversion Lemma in [**S0**]. The Inversion Lemma gives shape to the renormalization map R.

Central Tiles: When $s \in (1/2, 1)$, the intersection $F_1 \cap F_2$ is an octagon, which we call the *central tile*. When $s \leq 1/2$ or $s \geq 1$, the intersection $F_1 \cap F_2$ is a square. This square generates a grid in the plane, and finitely many squares in this grid lie in $X = F_1$. We call these squares the *central tiles*. See Figure 2.5.

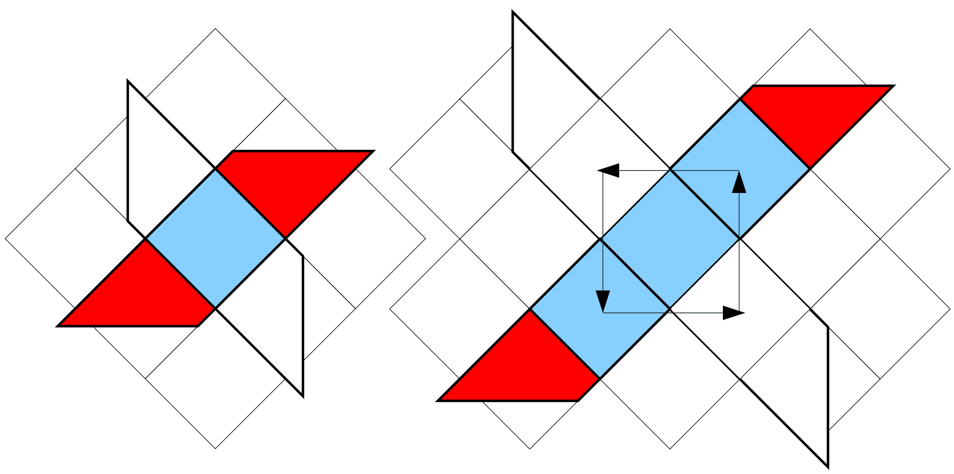

Figure 2.5: The central tiles (blue) for $s = 5/4$ and $t = 9/4$.

For any relevant object $S \subset X_s$, the set S^0 denotes the portion of S lying to the left of the central tiles of Δ_s. We will use this notational convention repeatedly.

2.6 Insertion Symmetry

When $s \geq 1$ and $t = s+1$, the intersections $\Delta_s^0 = \Delta_s \cap X_s^0$ and $\Delta_t^0 \Delta_t \cap X_s^0$ are isometric. (We give the argument in [**S0**].) Combining this fact with the inversion symmetry, we get the following result, also stated in [**S0**].

Lemma 2.6 (Insertion) *If $s < 1/2$ and $t = s/(2s+1)$ then $\Delta_s \cap X_s^0$ and $\Delta_t \cap X_t^0$ are similar.*

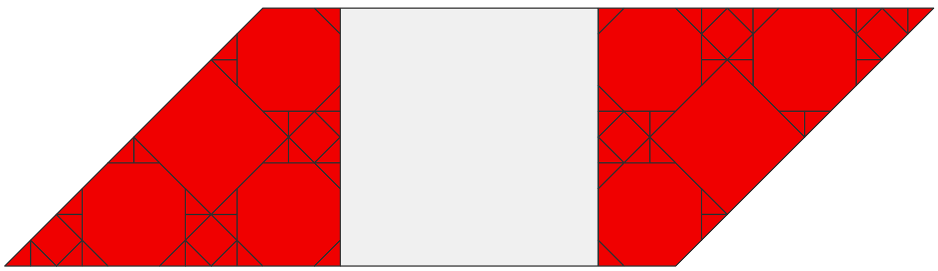

Figure 2.6: Δ_s for $s = 5/13$.

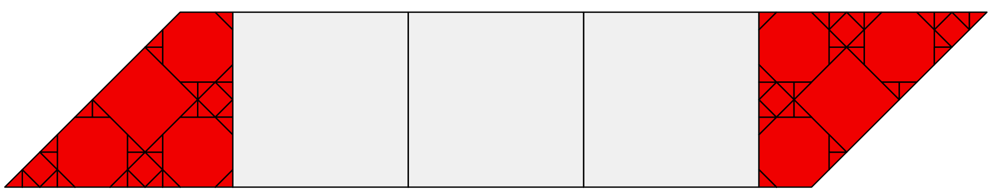

Figure 2.7: Δ_t for $t = 5/23$.

The reason for the name of this result is that the picture for t is obtained from the picture for s just by inserting 2 new central squares. The reader can see this in action by comparing Figures 2.6 and 2.7.

In view of the Insertion Lemma, we will often describe our results for $s \in [1/4, 1)$. The one other advantage of the Insertion Lemma, which we exploited in [**S0**] is that often a statement for parameters in $[1/4, 1)$ involves a finite computational proof whereas the statement for all parameters in $(0, 1)$ would require (if computed directly) an infinite computation.

2.7 Bilateral Symmetry

The kind of symmetry described in this section looks obvious from the pictures, but it required a nontrivial computational proof in [**S0**]. We will describe the symmetry for $s \in [1/4, 1)$.

We say that a line L is a *line of symmetry* for Δ_s if

$$\Delta_s \cap \Big(X_s \cap R_L(X_s)\Big) \tag{5}$$

is invariant under the reflection R_L in L. Note that X_s itself need not be invariant under R_L. Consider the following lines.

- Let H be the line $y = 0$.
- Let V be the line $x = -1$.
- For D_s be the line of slope -1 through $(-s, -s)$.
- For $s \in [1/4, 1/2]$, E_s be the line of slope -1 $(-3s, -3s)$.
- For $s \in [1/2, 1]$, E_s is the line of slope 1 through $(-s, -s)$.

We call these the *fundamental lines of symmetry*. For each line L above, the line $\iota(L)$ is also a line of symmetry. However, we will not usually refer to these other lines.

H, V, and D_s and E_s respectively are the lines of symmetry for

$$A_s = (X_s \cap R_H(X_s))^0, \qquad B_s = X_s \cap R_V(X_s),$$

$$P_s = X_s \cap R_D(X_s), \;\; Q_s = X_s \cap R_D(X_s). \tag{6}$$

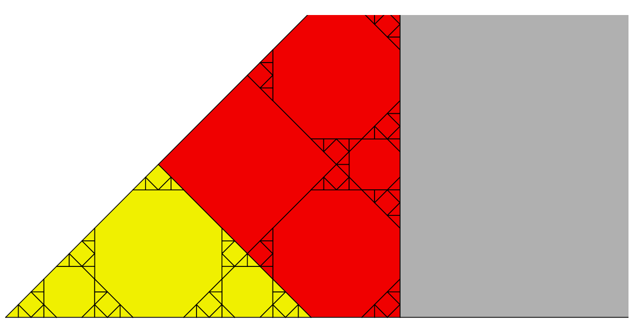

Figure 2.8: A_s (red) and B_s (yellow) for $s = 12/31$.

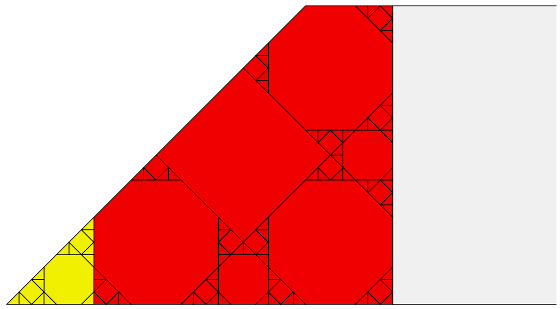

Figure 2.9: P_s (red) and Q_s (yellow) for $s = 12/31$.

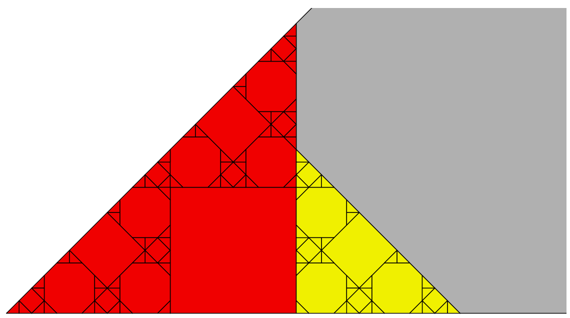

Figure 2.10: P_s (red) and Q_s (yellow) for $s = 18/23$.

Remarks:
(i) In case $s \in (1/2, 1)$, the bilateral symmetry behaves nicely with respect to the inversion symmetry. The value $r = 1/(2s)$ also lies in $(1/2, 1)$. There is a similarity carrying Δ_r to Δ_s, and this similarity carries the regions (A_r, B_r, P_r, Q_r) to the regions (Q_s, P_s, B_s, A_s). So, for $s \in (1/2, 1)$, the symmetries associated to (P_s, Q_s) are equivalent to the ones associated to (A_s, B_s). This is not true for $s < 1/2$.
(ii) Looking at the figures, the reader will probably be able to find other regions of bilateral symmetry. I think that all the bilateral symmetry one sees in the picture is a consequence of the ones already mentioned, together with renormalization.
(iii) In [**S0**] we defined A_s to be the hexagon $X_s \cap R_H(X_s)$, but here it seems better just to take the portion of this hexagon which lies to the left of the central tiles of Δ_s. This notation change causes no troubles.
(iv) The sets A_s, B_s, P_s, Q_s, which we call *symmetric sets*, are all clean. This follows from the fact that each side of one of these sets either lies in ∂X_s or can be moved to ∂X_s by the bilateral symmetry.

2.8 Squares in the Tiling

Here we sketch the proof of a result from [**S0**] which illustrates some of the power of the results mentioned above.

Lemma 2.7 *When s is rational and oddly even, Δ_s contains only squares and right-angled isosceles triangles. When s is irrational and oddly even, Δ_s contains only squares.*

Proof: When $s = 1/2$ we check that Δ_s consists entirely of squares and right-angled isosceles triangles. From the Insertion Lemma, the same result holds for $s = 1/(2n)$ for $n = 2, 3, 4...$ In general, the result follows from induction on the length of the orbit $R^n(x)$ and Theorem 2.5. In [**S0**] we showed that the triangles vanish in the irrational limit. So, when s is irrational and oddly even, Δ_s consists only of squares. ♠

Δ_s contains two kinds of squares. We say that a *box* is a square whose sides are parallel to the coordinate axes. We say that a *diamond* is a square whose sides have slope ± 1. Then Δ_s consists entirely of boxes and diamonds. This follows from induction and the same kin dof proof given in Lemma 2.7.

3 The Pattern of Filling

3.1 The First Half

In this chapter we elaborate on Theorem 2.5, explaining more precisely how Δ_s is composed of parts of $\phi_s(\Delta_t \cap Y_t)$. Here $t = R(s)$ and ϕ_s are as in Theorem 2.5.

We first consider the case when $s < 1/2$. Let $t = R(s)$ and $u = R(t)$.

- If $t > 1/2$ let $K = 1$.
- If $t < 1/2$ and $u < 1/2$, let $K = \text{floor}(1/(2t))$.
- If $t < 1/2$ and $u > 1/2$, let $K = 1 + \text{floor}(1/(2t))$.'

The number $K = K(s)$ plays an important role in our constructions. We call it the *layering constant*

Let U_s be the image, under of ϕ_s, of the trivial tile in Δ_t. This is the dark red tile in Figures 3.1 and 3.2 below. Define

$$\Psi_s^0 = Z_s^0 \cup U_s. \tag{7}$$

Let τ_s denote the subset of Z_s^0 lying beneath the line extending the top right edge of U_s. Here τ_s is the colored region in Figures 3.1 and 3.2.

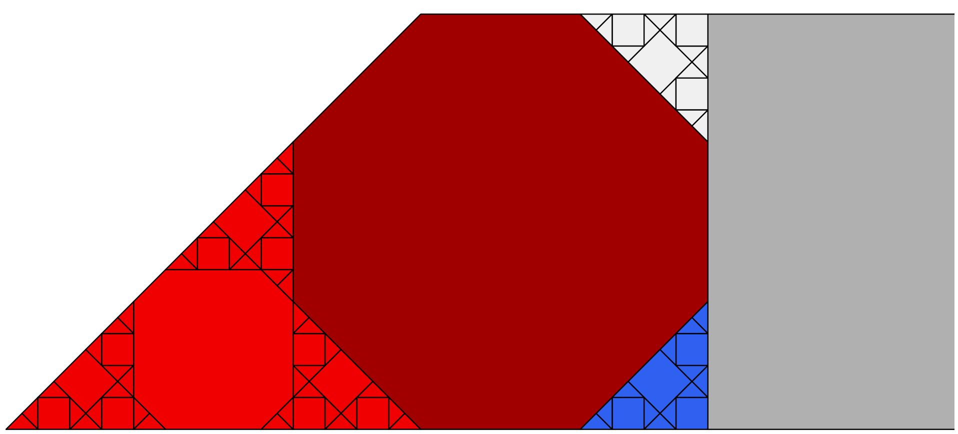

Figure 3.1: Ψ_s^0 (red) and U_s (dark red) and τ_s (red, blue) for $s = 13/44$.

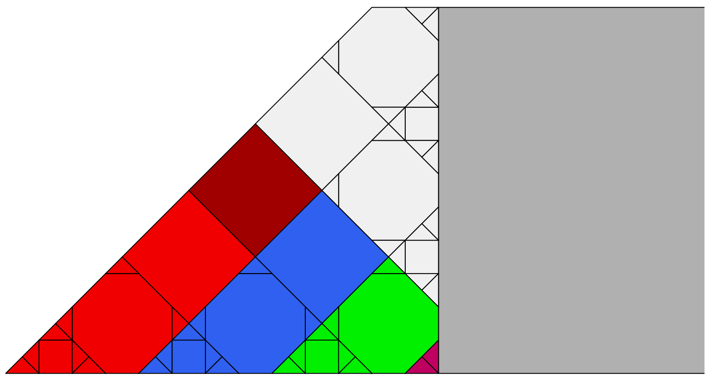

Figure 3.2: Ψ_s^j for $j = 0, 1, 2, 3$ (red, blue, green, magenta) for $s = 11/26$.

Let T_s denote the transformation which translates by a vector pointing in the positive x direction and having length equal to the length of the bottom side of Ψ_s^0. Define

$$\Psi_s^j = T_j(\Psi_s^0) \cap \tau_s. \tag{8}$$

We have

$$\tau_s = \bigcup_{j=0}^{K} \Psi_s^j. \tag{9}$$

For larger j, the sets in Equaation 9 are empty.

The whole tiling is determined by the tiling inside τ_s and symmetry.

$$X_s = X_s^0 \cup \text{central tiles} \cup \iota(X_s^0), \qquad X_s^0 = \tau_s \cup R_D(\tau_s). \tag{10}$$

The second equation is a consequence of th fact that the top right boundary of τ_s is parallel to the fundamental symmetry line D_s and lies above it.

The following result explains the structure of Δ_s inside τ_s.

Lemma 3.1 (Filling) *$T_s^{-j}(\Delta_s \cap \Psi_s^j) = \Delta_s \cap T^{-j}(\Psi_s^j)$ for all $j = 1, ..., K$.*

Proof: What the result means is that the map T_s^{-j} respects the tiling, at least on the relevant domain.

By the Insertion Lemma, we can take $s \in (1/4, 1/2)$. Referring to the basic map of our system, the map $f_s : X_s \to X_s$ is defined in terms of a partition of X_s. On each piece of this partition, f_s is a translation. Let Ω_s be the piece of the partition which shares the lower left vertex of X_s^0. A routine calculation shows that the restriction of f_s to Ω_s is T_s.

When $s \in (1/3, 1/2)$, the set Ω_s is the quadrilateral with the following properties.

- The left edge of Ω_s is contained in the left side of X_s.
- The bottom edge of Ω_s is contained in the bottom edge of X_s.
- The top edge of Ω_s lies in the same line as the top edge of Z_s^0.
- The right edge e_s of Ω_s is vertical and has the property that $T_s(e)$ lies in the left edge of the leftmost central tile of Δ_s.

When $s \in (1/4, 1/3)$, the set Ω_s is a triangle whose left, right, and bottom sides are as above. Our argument works the same in either case.

Figures 4.3-4.5 illustrate the picture for $s = 19/60, 11/30, 9/20$. The diagonal line L_s bisects the yellow square and the green diamond in each picture.

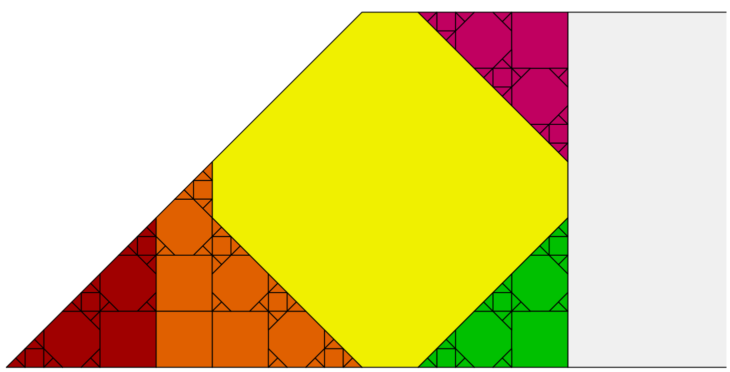

Figure 3.3: Z_s^0 (red, orange), Ω_s (red) and $f_s(\Omega_s)$ green

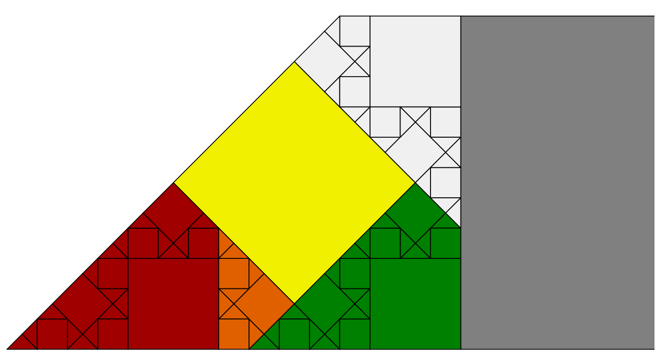

Figure 3.4: Z_s^0 (red, orange) and Ω_s (red) and $f_s(\Omega_s)$ (green).

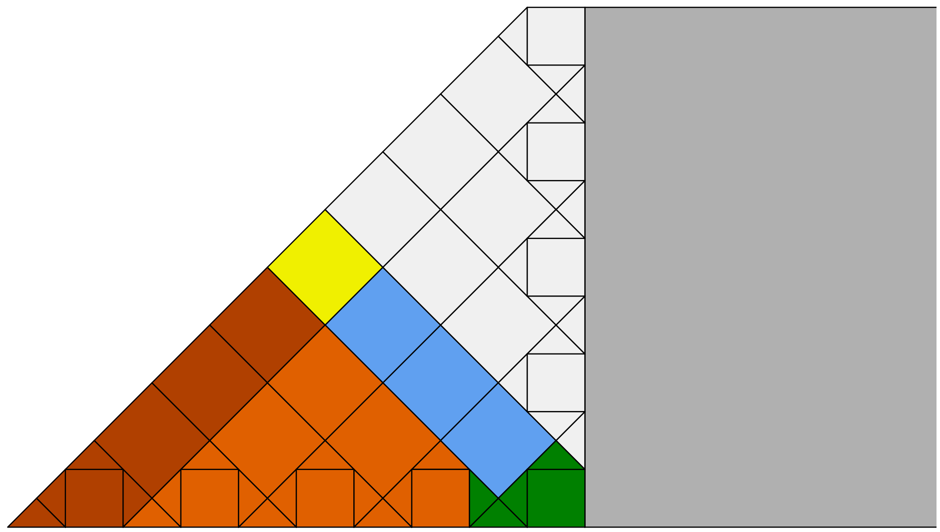

Figure 3.5: Z_s^0 (red), Ω_s (red, orange), and $f_s(\Omega_s)$ (green, orange, blue).

A routine calculation shows that

$$T_s^{-1}(\tau_s - \Psi_s^0) \subset \Omega_s. \tag{11}$$

If we start with any point $p \in \tau_s - \Psi_s^0 = Z_s^0 \cup U_s$, we see from the shape of Ω_s^0 that the iterates $f_s^{-j}(p)$ are defined and lie in Ω_s^0, for each $j = 0, 1, 2, ...$ until we reach some $k \leq K$ such that $f_s^{-k}(p) \subset Z_s^0$. Our result follows from this observation and from the fact that Δ_s is f_s-invariant. ♠

Say that a *central tile of* $\Delta_s \cap \Psi_s^0$ is the image of a central tile of Δ_t under the map ϕ_s. For instance, in Figure 3.5, there are 4 central tiles, one yellow and 3 dark red. The central tiles all have the same size, and lie in Ψ_s^0.

For $j < K$, the tiling $\Delta_s \cap \Psi_s^j$ has a simple description.

- Start with the tiling $\Delta_s \cap \Psi_s^0$.
- Chop off the top j central tiles.
- Translate by T_s^j.

The tiling $\Delta_s \cap \Psi_s^K$ is more complicated, but we do not need to know it explicitly.

3.2 The Second Half

Now we explain the picture for $s \in (1/2, 1)$. This time, we define

$$K = \text{floor}\Big(\frac{1}{2-2s}\Big) \tag{12}$$

Again, we call K the layering constant.

This time, the right edge of Z_s^0 lies on the same line as the left edge of the central tile of Δ_s. Let δ_s denote this line. Let T_s denote the translation by the vector which is positivel proportional to $(1,1)$ and whose length is the same as the length of the left side of Z_s^0. Let τ_s be the region of X_s^0 lying to the left of δ_s. In Figure 3.6, the region τ_s is colored blue/red/yellow.

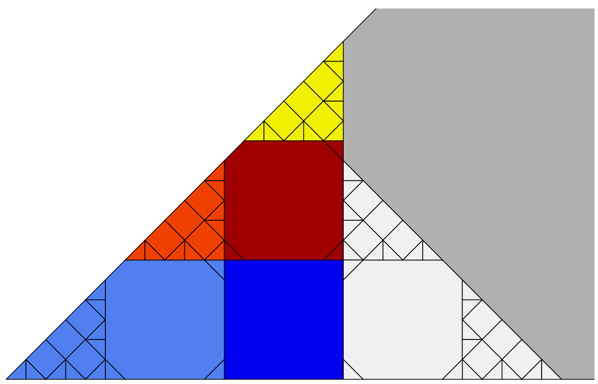

Figure 3.6: Ψ_s^j for $j = 1, 2, 3$ (red, blue, yellow) for $s = 14/17$.

Define

$$\Psi_s^j = T_s^j(Z_s) \cap \tau_s. \tag{13}$$

We have

$$\tau_s = \bigcup_{j=0}^{K} \Psi_s^j. \tag{14}$$

This time we have

$$X_s = X_s^0 \cup \text{central tiles} \cup \iota(X_s^0), \qquad X_s^0 \subset \tau_s \cup R_V(\tau_s), \tag{15}$$

where V is the vertical line of symmetry, $x = -1$.

With these definitions, the Filling Lemma holds *verbatim*, and the proof is essentially the same. This time Ω_s is a right isosceles triangle, and the left edge of $f_s(\Omega_s$ lies in δ_s, and f_s translates diagonally along the vector that generates the left side of Z_s. In Figure 3.6, Ω_s is the union of the light red and light blue tiles.

3.3 Pyramids

Now we explain more of the structure. We will concentrate on the case $s < 1/2$.

Let $K = K(s)$ be the layering constant. Say that a *pyramid* of size k is a configuration of diamonds having the structure indicated in Figure 3.7 for $k = 1, 2, 3$. We refer to the longest (diagonal) row of diamonds as the *base* if the pyramid.

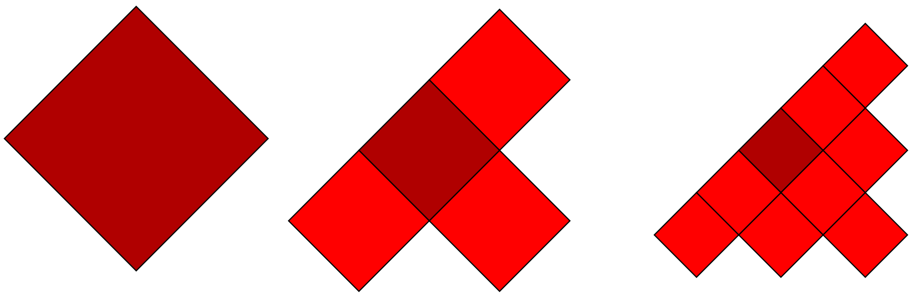

Figure 3.7: Pyramids for $k = 1, 2, 3$.

Figure 3.8 shows a red pyramid of size 2 contained in Δ_s for $s = 30/73$. In the next section, we will explain the meaning of the blue squares shown in the figure.

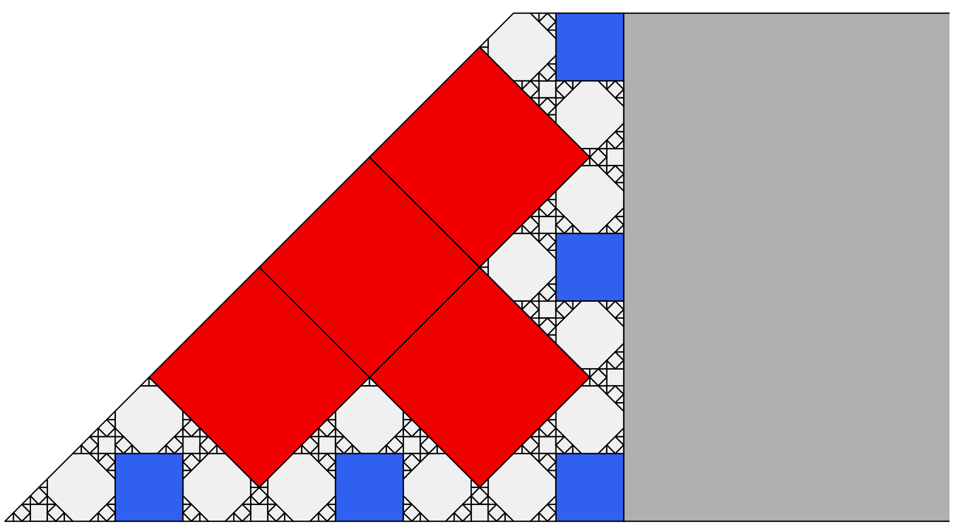

Figure 3.8: The pyramid. $s = 30/73$, $R(s) = 13/60$ and $R^2(s) = 4/13$.

Say that an *extended pyramid* of size K is the union of polygons obtained by taking the outer squares in each row and chopping off the corners so as to leave semi-regular octagons. Figure 3.9 shows an example of an extended pyramid of size K.

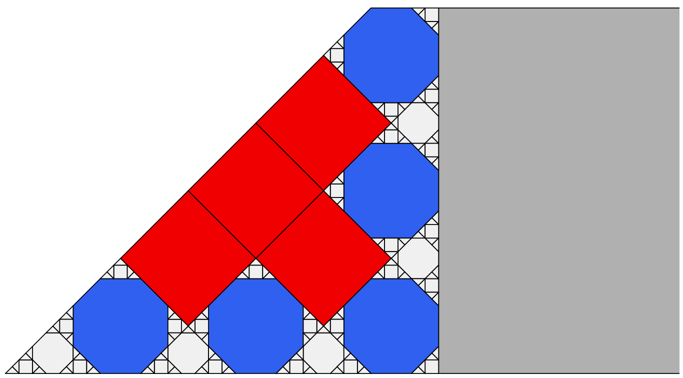

Figure 3.9: Extended Pyramid. $s = 27/64$, $R(s) = 5/27$ and $R^2(s) = 7/10$.

Say that $s < 1/2$ has *type 0* if $R(s) < 1/2$ and $R^2(s) < 1/2$. Otherwise, say that s has *type 1*.

Lemma 3.2 *Suppose s has type* 0 *(respectively type* 1*). Then Δ_s contains an ordinary (respectively extended) pyramid of size K.*

Proof: Suppose first that $R^2(s) < 1/2$. The bottom K squares in the base of the desired pyramid are guaranteed by Theorem 2.5 and the Insertion Lemma (applied to Δ_t). The bottom half of the pyramid is then guaranteed by the Filling Lemma. The top half is then guaranteed by the bilateral symmetry corresponding to the diagonal line D_s.

The proof is essentially the same when $R^2(s) > 1/2$. What happens here is that the tiles in Δ_t just to the left of the leftmost central tile is an octagon which has the same height as the central tiles. have the same width as the central tiles. Here $t = R(s)$. If we simply keep track of this additional octagon, we get the same structure as in the other case, except that these octagons replace the outer layer of diamonds. ♠

Remark: One has a result similar to Lemma 3.2 when $s > 1/2$. When $s > 1/2$ and $R^2(s) < 1/2$ we get an ordinary pyramid of size $K - 1$. When $s > 1/2$ and $R^2(s) > 1/2$ we get an extended pyramid of size K.

3.4 The Octagrid

Remark: The reader only interested in Statement 1 of the Main Theorem can skip the rest of this chapter.

In this section, we consider the case when $s < 1/2$ and $R(s) < 1/2$. One can do something similar for other parameter ranges, but we do not need the construction otherwise.

First we consider the case $R^2(s) > 1/2$. Thanks to the equalities between the widths, the distance between the center of an octagon and the center of an adjacent square in the extended pyramid is the same as the distance between two adjacent squares. Put another way, the union of the centers of the tiles in the extended pyramid is contained in a square grid.

When $R^2(s) < 1/2$, there is a similar phenomenon. This is where the blue squares in Figure 3.8 (and Figure 3.10) come in. These squares have the following description. The bottom left blue square is the image, under ϕ_s, of the central tile of Z_t. The remaining blue tiles exist as a consequence of the Filling Lemma and bilateral symmetry. We call the union of the pyramid and these blue squares the *extended pyramid*. The union of the centers of the tiles of the extended pyramid, in this case also, is part of a square grid. This follows from a routine calculation, which we omit.

In either case, we form the *octagrid* by taking the union of horizontal, vertical, and diagonal (meaning slope=± 1) lines through the centers of the tiles of the extended pyramid. The octagrid chops up Δ_s into what turn out to be usefully small pieces.

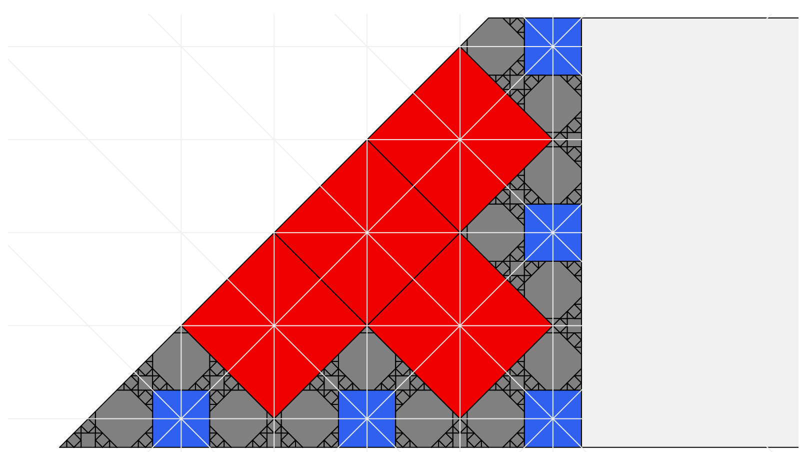

Figure 3.10: The octagrid for $s = 30/73$.

There are finitely many lines in the octagrid. These lines partition X_s^0 into finitely many bounded convex regions. We call these regions *octagrid components*. Each octagrid component is some open convex polygon.

Lemma 3.3 *Let G be any octagrid component. There is an isometry carrying $\Delta_s \cap G$ to a subset of $\Delta_s \cap Z_s^0$.*

Proof: The fundamental line D_s of symmetry is contained in the octagrid. The reflection R_d carries each octagrid component above D into some octagrid component below D. Hence, by symmetry, it suffices to prove our result when G lies beneath D_s.

G lies in the region τ_s from the Filling Lemma. Applying the map T_s^{-1} from Equation 8 as many times as we can, we can assume that (the interior of) G intersects Z_s^0. If $G \subset Z_s^0$, we are finished. If G is contained in one of the squares of the pyramid, we are finished.

The only remaining possibility is that G has a vertex on the centerline of Z_s^0 (the line $\phi_s(H)$) but extends over the right edge of Z_s^0. The problem is that the diagonal edge of G is perpendicular rather than parallel to the centerline of Z_s^0. To fix this problem, we consider $G' = T_s^j \circ R_V(G)$ for the largest value of j we can take. The interior of G' intesects Z_s^0 and the diagonal edge of G' is parallel to Z_s^0. Hence, $G' \subset Z_s^0$. Since we have only applied symmetries of the tiling, we see that $\Delta_s \cap G$ and $\Delta_s \cap G'$ are isometric. ♠

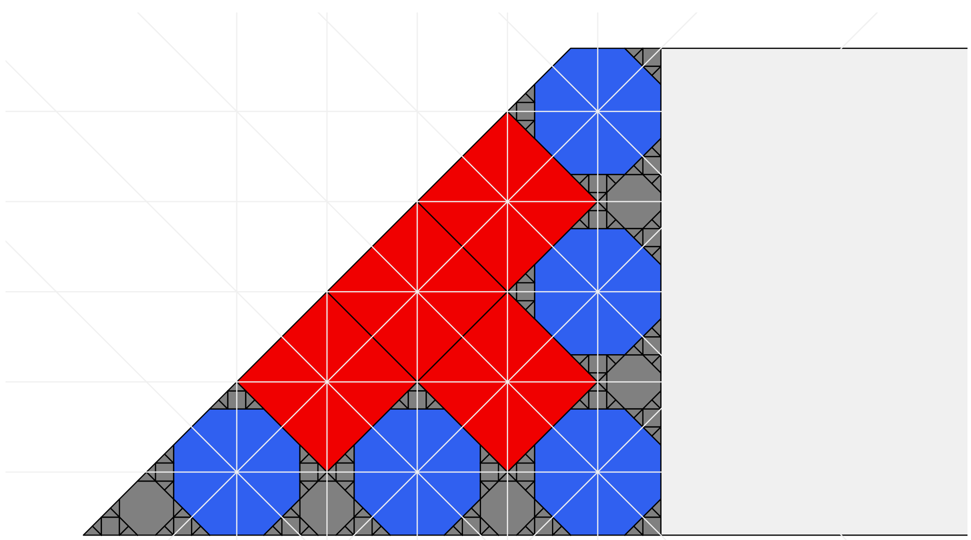

Figure 3.11: The octagrid for $s = 27/64$.

4 Geometric Lemmas

In this chapter we prove 3 geometric lemmas, which we call the Shield Lemma, the Pinching Lemma, and the Covering Lemma. Only Statement 3 of the Main Theorem requires the Covering Lemma.

4.1 The Shield Lemma

Let A_s be the symmetric piece from §2.7.

Lemma 4.1 (Shield) *Let s be irrational and oddly even. Every point of ∂A_s, except the 2 vertices having obtuse angles, is contained in the edge of square. Those points which belong to the boundaries of more than one tile are the vertices of pairs of adjacent squares.*

Figure 4.1 illustrates the Shield Lemma. When s is rational, there are 2 small triangles touching the obtuse vertices of A_s. These triangles vanish in the irrational limit.

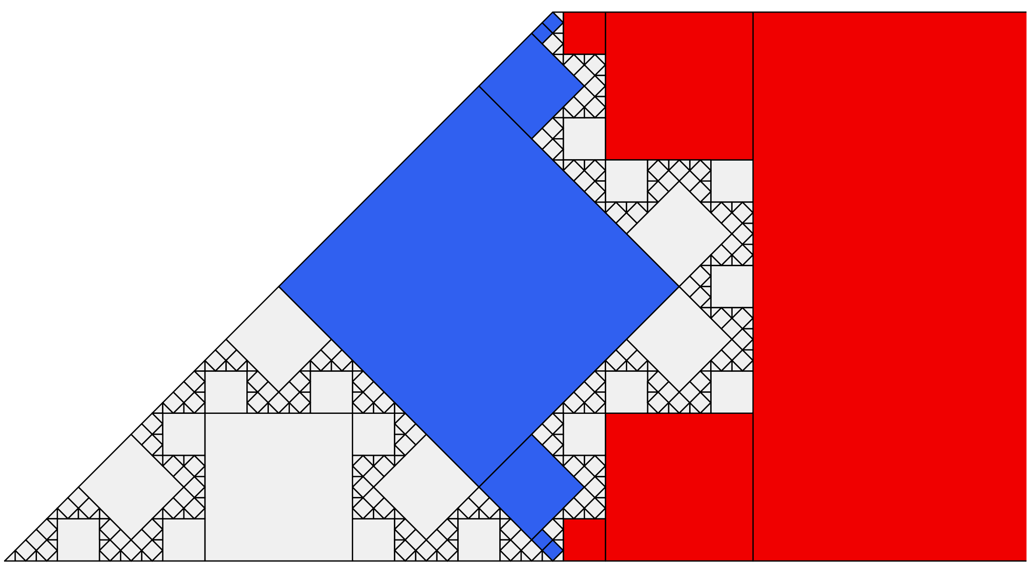

Figure 4.1: A_s for $s = 26/71$.

We define the *shield* Σ_s to be the union of the top left edge of A_s and the left half of the top edge. Σ_s is the union of two line segments. The top left vertex ν_s of X_s (and A_s) is the place where the two line segments join.

By symmetry, it suffices to prove the Shield Lemma for the points of ∂A_s contained in the shield. We analyze the picture in the rational case and then take limits. In this section we work out how Δ_s sits in A_s when s is rational and oddly even.

We say that a square of Δ_s *abuts* Σ_s if $\overline{T} \cap \Sigma_s$ is nonempty. In this case, the intersection is a line segment which we call the *contact* between the square and the shield. The *radius* of T is the distance from the center of T to a corner of T. We call a radius ρ *realized*, if a square of Δ_s having radius ρ abuts the shield.

Lemma 4.2 *Let s be an oddly even rational. The following is true.*

1. *Exactly one triangle of Δ_s abuts Σ_s and the segment of contact contains ν_s as an endpoint.*
2. *The squares which abut Σ_s occur in monotone decreasing size, largest to smallest, as one moves from an endpoint of Σ_s to ν_s.*
3. *The number of squares of each size is determined by the even expansion of s.*
4. *Let ρ be a realized radius. Some square of radius ρ, which abuts Σ_s, has a vertex within ρ of ν_s.*

Let us assume Lemma 4.2 for now, and finish the proof of the Shield Lemma. Let $\{r_n\}$ be a sequence of oddly even rationals which converges to s. Given the convergence of tilings described above, we see that the union of square tiles abutting Σ_s is the Hausdorff limit of the union of square tiles abutting Σ_{r_n}, as $n \to \infty$. The size of the single triangle in the picture for the rational parameters tends to 0. Hence, every point of $\Sigma_s - \nu_s$ is contained in a segment of contact for some square. The main point to worry about is that somehow there is a point $p \in \Sigma_s - \nu_s$, with the following property: As n tends to ∞, the square whose segment of contact contains p tends to 0 in size. This unfortunate situation cannot occur because it would violate Item 4 of Lemma 4.2.

Proof of Lemma 4.2: The proof goes by induction on the length of the orbit $\{R^n(s)\}$. When $s = 1/2$ the result holds by inspection. The case $s = 1/2n$ follows from the Insertion Lemma. For $s \neq 1/2n$, let $t = R(s)$. By induction, all the properties of the lemma hold for Δ_t.

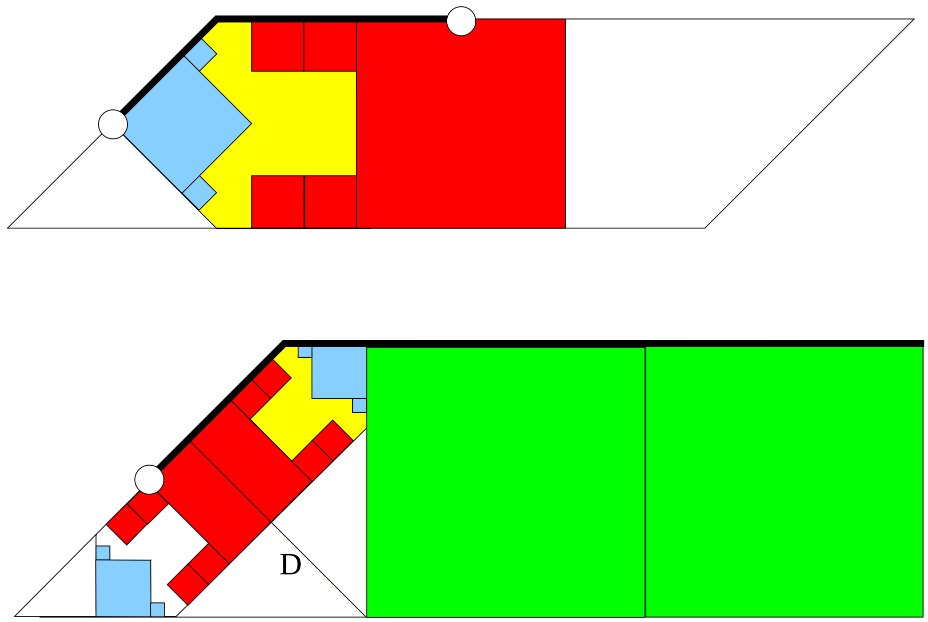

Figure 4.2: Inherited structure

Let $D = D_s$ be the diagonal line of bilateral symmetry and let R_D denote reflection in D. The yellow region at the top left of Figure 4.2 is A_t. The big blue square in the top of Figure 4.2, which is the central square of Ψ_t^0, is mapped by $R_D \circ \phi_s$ to a square which abuts the (green) leftmost central square of Δ_s.

The pattern of tiles abutting the shield Σ_s is the same as the pattern of squares abutting the shield Σ_t, except that some green ones have been appended. The number of green ones is determined by the number n_0 in the even expansion of s. All the points in our lemma follow from this structure. ♠

Corollary 4.3 *Let S_s denote the left half of $\widehat{\Lambda}_s$. There exists a convex set D_2 such that $S_s \cap \text{interior}(A_s) \subset D_2$ and D_2 intersects ∂A_s only at the two obtuse vertices.*

Proof: We simply slice off from A_s suitably chosen neighborhoods of the square tiles which abut the edges of A_s. With a little care (i.e., by making these neighborhoods shrink very rapidly as we approach the vertices) we can make the resulting set convex. ♠

4.2 The Pinching Lemma

Let S_s denote the left half of the limit set $\widehat{\Lambda}_s$.

Lemma 4.4 (Pinching) *At most one point of S_s lies on each fundamental line of bilateral symmetry.*

We argue by contradiction. Let $g(s)$ denote the diameter of X_s^0. Say that a *counterexample* is a quadruple $\Omega = (L, p_1, p_2, s)$, where $L = L_s$ is a fundamental line of symmetry and $p_1 \neq p_2 \in L \cap S_s$. We call s the *parameter* of the counterexample. We define

$$\lambda(\Omega) = \frac{\|p_1 - p_2\|}{g(s)}. \tag{16}$$

If Ω' is another counterexample obtained from Ω using either the Insertion Lemma or (when $s > 1/2$) the Inversion Lemma, we have $\lambda(\Omega) = \lambda(\Omega')$. Moreover

$$\lambda = \sup_{\Omega} \lambda(\Omega) \leq 1 \tag{17}$$

Lemma 4.5 *For any $\epsilon > 0$ there exists a counterexample Ω, whose parameter is less than $1/2$, such that $\lambda(\Omega) > \lambda - \epsilon$.*

Proof: Certainly, there exists a counterexample $\Omega = (L, p_1, p_2, s)$ such that $\lambda(\Omega) > \lambda - \epsilon$. If $s < 1/2$ we are finished. If $s > \sqrt{2}/2$ we can apply the Inversion Lemma to reduce to the case $s \in (1/2, \sqrt{2}/2]$. In this case, we have $K = 1$, where K is the layering constant.

Let $t = R(s) = 1 - s$. When $K = 1$ we have the following facts.

- $R_D \circ \phi_s(B_t) = A_s$ (red) is a neighborhood of H in A_s.
- $R_D \circ \phi_s(A_t)$ (green) is a neighborhood of V in B_s.
- $\phi_s(P_t)$ (light green) is a neighborhood of D_s in P_s.
- $R_V \circ \phi_s(Q_t)$ (yellow) is a neighborhood of E_s in Q_s.

The colors refer to the various regions in Δ_s shown in Figure 4.3. In all cases, we can use the relevant map to pull back the counterexample isomerically to Δ_t, and we get a counterexample involving t with the same diameter. But then the λ-value of this counterexample does not decrease. Hence, the new counterexample Ω' satisfies $\lambda(\Omega') > \lambda - \epsilon$ and has parameter $t < 1/2$. ♠

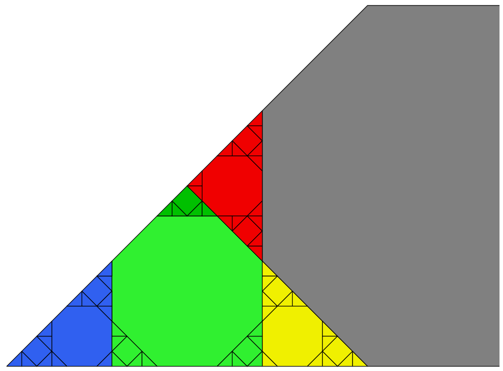

Figure 4.3: Various regions in Δ_s for $s = 12/17$.

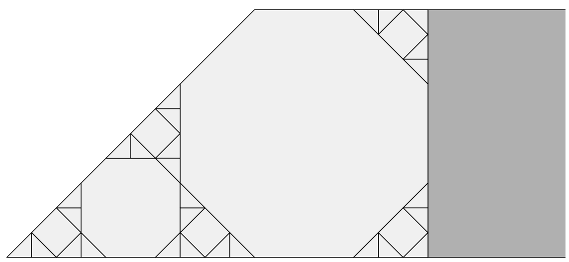

Figure 4.4: The left half of Δ_t for $t = R(s) = 5/17$.

By the previous result, and the Insertion Lemma, we can find a counterexample Ω having parameter $s \in (1/4, 1/2)$ such that $\lambda(\Omega)$ is as close as we like to λ.

Consider a counterexample Ω having $s \in (1/4, 1/3)$. For s in this range, we have $t = R(s) > 1/2$. The blue octagon in Δ_s is the image under ϕ_s, of the trivial tile in Δ_t. Call this octagon O_s.

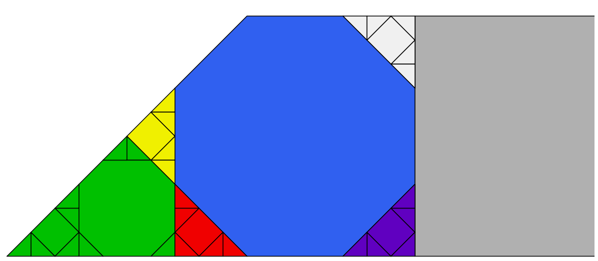

Figure 4.5: O_s (blue) and other regions for for $s = 5/17$.

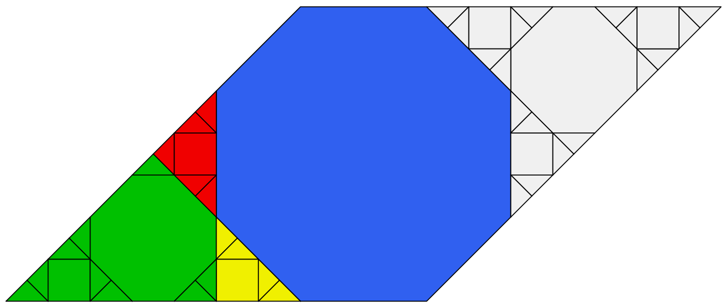

Figure 4.6: Δ_t for $t = R(5/17) = 7/10$.

We have the following facts.

- $\phi_s(Q_t)$ (yellow) is a neighborhood of $H^0 - O_s$.
- $\phi_s(P_t) = B_s$ (red, green).
- $R_H \circ R_D \circ \phi_s(A_t)$ (purple) is a neighborhood of $D_s - O_s$.
- $\phi_s(B_t) = Q_s$ (green, yellow).

The colors refer to Figures 5.4 and 5.5. In all cases, we can pull the counterexample back by the relevant similarity to get a counterexample associated to the parameter t. Call the new counterexample Ω'. Since $Y_t^0 = X_t^0$ for $t > 1/2$, we have

$$\lambda(\Omega') = \frac{\|\phi_s^{-1}(p) - \phi_s^{-1}(q)\|}{\mathrm{diam}(Y_t^0)} = \frac{\|p - q\|}{\mathrm{diam}(\phi_s(X_t^0))} =$$

$$\frac{\|p - q\|}{\mathrm{diam}(Z_s^0)} = \lambda(\Omega)\frac{\mathrm{diam}(X_s^0)}{\mathrm{diam}(Z_s^0)} = K_s\lambda(\Omega). \tag{18}$$

Here the constant $K_s > 1$ is uniformly bounded away from 1, by compactness. But this is a contradiction. We could choose $\lambda(\Omega) > \lambda/K_s$ and then we would have $\lambda(\Omega') > \lambda$.

Now consider a counterexample with $s \in (1/2, 1/3)$. The argument is very similar to what we just did. This time, we let O_s be the pyramid from Lemma 3.2. In this case, the same remarks apply about the points of the counterexample being disjoint from (the interior of) O_s. We have the following facts.

- $R_D \circ T_s^j \circ \phi_s(P_t)$ (red) is a neighborhood of $H^0 - O_s$ for a suitable choice of j. Here T_s^j is as in Equation 8.

- $T_s^j \circ \phi_s(Q_t)$ (blue) is a neighborhood of $V - O_s$.
- $R_V \circ \phi_s(A_t^0)$ (yellow) is a neighborhood of $D - O_s$.
- $E_s = \phi_s(B_t)$ (green).

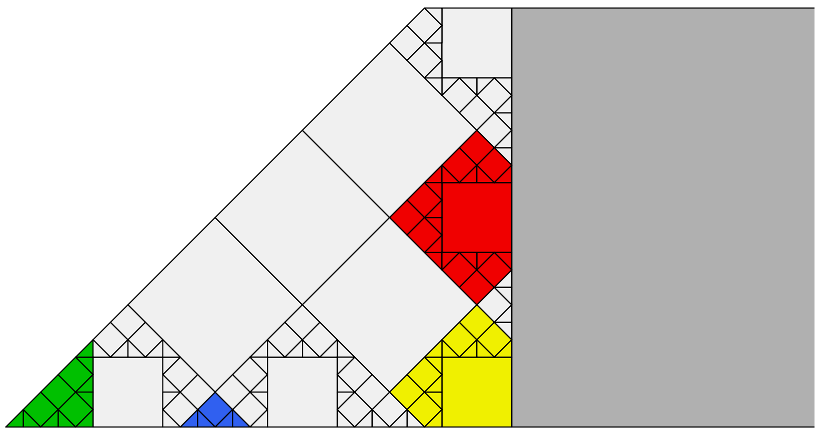

Figure 4.7: Some regions of Δ_s for $s = 12/29$.

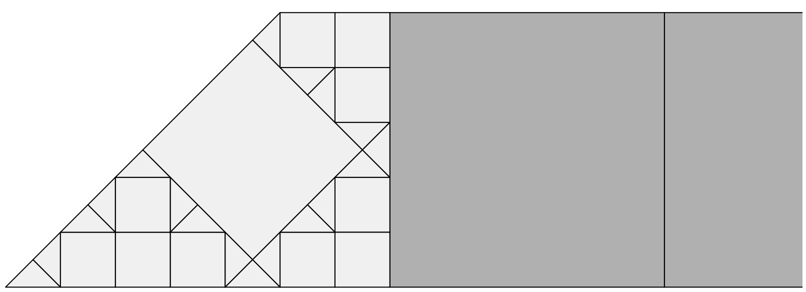

Figure 4.8: The left half of Δ_t for $t = R(12/29) = 5/24$.

This time, we get

$$\lambda(\Omega') = \lambda(\Omega)\frac{\mathrm{diam}(X_s^0)}{\mathrm{diam}(\phi_s(X_t^0))} = L_s\lambda(\Omega) \tag{19}$$

Here $L_s > 1$ is uniformly bounded away from 1. We get the same contradiction as before. This completes the proof of the Pinching Lemma.

4.3 The Covering Lemma

We say that an ϵ-*patch* is a triple (K,ψ,u) where

- $u \in (0,1)$.
- K is one of the 4 symmetric sets A_u, B_u, P_u, Q_u from §2.7.
- $\psi : K \to X_s$ is a similarity which contracts by some factor $\lambda \leq \epsilon$.
- $\psi(\Delta_u \cap K) = \Delta_s \cap \psi(K)$.

The last condition means that ψ gives a bijection between tiles of $\Delta_u \cap K$ and tiles of $\Delta_s \cap \Psi(K)$. When we have an ϵ-patch (K,ψ,u), we are recognizing a small portion of Δ_s as being a similar copy of a large portion of Δ_u. When the choice of ϵ is not relevant to the discussion, we will just say *patch* in place of ϵ-patch.

We present two versions of our result. In these versions, the constants m and n depend on everything in sight. They are meant to (possibly) vary from case to case. Here is the first version.

Lemma 4.6 *Suppose $s \in (0,1)$ is irrational, and let $t = R(s)$. Then for each set $K_s \in \{A_s, B_s, P_s, Q_s\}$, we have*

$$K_s = \bigcup_{i=1}^{m} \alpha_i \cup \bigcup_{j=1}^{n} \beta_j,$$

where α_i is a tile of Δ_s and β_j is the image of some ϵ-patch (K_j,ψ_j,t). Here ϵ is the scale factor of ϕ_s, the map from Theorem 2.5.

Iterating Lemma 4.6 for the sequence $\{R^n(x)\}$ and using the fact that $X_s^0 = A_s \cup B_s = P_s \cup Q_s$, we get the following corollary.

Corollary 4.7 (Covering) *For any $\epsilon > 0$ we have*

$$\Delta_s = \bigcup_{i=1}^{m} \alpha_i \cup \bigcup_{j=1}^{n} \beta_j, \qquad (20)$$

where α_i is a tile of Δ_s and β_j is the image of some ϵ-patch (K_j,ψ_j,u). Moreover, we can take $u = R^k(s)$ for any sufficiently large s. Equation 20 likewise holds for $\Delta_s \cap K_s$ in place of Δ_s for each symmetric piece K_s.

Lemma 4.6 has the same kind of proof as the Pinching Lemma. Let $K = K(s)$ be the layering constant for s which appears in the Filling Lemma.

Lemma 4.8 *Lemma 4.6 is true when $s > 1/2$ and $K(s) > 1$.*

Proof: Figure 4.9 illustrates a typical picture for s in this range.

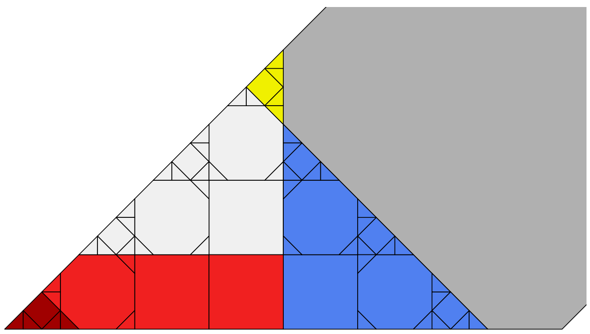

Figure 4.9: The tiling Δ_s^0 for $s = 13/15$. Here $K(s) = 3$.

We have the following equations.

- $A_s = R_D \circ \phi_s(B_t)$. A_s is colored yellow in Figure 4.9, and $\phi_s(B_t)$ is colored dark red. R_D is reflection in the diagonal line $D = D_s$ of symmetry.

- $$B_s = \nu \cup R_V(\nu) \cup R_D \circ \psi_s \circ (A_t) \cup \Theta, \qquad \nu = \bigcup_{k=0}^{K-1} \Psi_s^k.$$

  The set B_s is colored red/white/blue in Figure 4.9. Here V is the vertical line of symmetry, and Θ is a finite union of square tiles.

- $P_s = \nu \cup R_D(\nu)$. In Figure 4.9, P_s is colored red/white/yellow.

- $$Q_s = R_D(\nu'), \qquad \nu' = T_s^{-1}\Big(\bigcup_{k=1}^{K-1} \Psi_s^k\Big).$$

  Here T_s is as in Equation 8 and Q_s is colored blue in Figure 4.9.

We can interpret all these equations as the desired patch coverings. ♠

Lemma 4.9 *Lemma 4.6 is true when $s > 1/2$.*

Proof: It only remains to treat the case when $K(s) = 1$. Figure 4.10 shows a representative picture for s in this range.

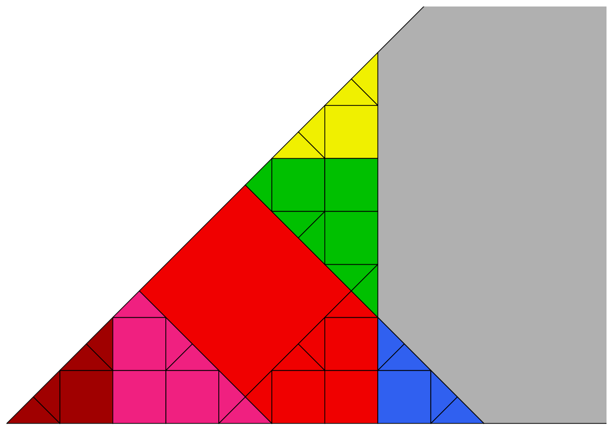

Figure 4.10: The tiling Δ_s^0 for $s = 9/14$. Here $K(s) = 1$.

- $A_s = R_D \circ \phi_s(B_t)$. (This is as above.) In Figure 4.10, A_s is colored yellow/green, and $\phi_s(B_t)$ is colored dark red/pink.

- $$B_s = R_d \circ \phi_s(A_t) \cup \phi_s(Q_t) \cup R_V \circ \phi_s(Q_t).$$

  B_s is the union of all the pieces not colored [3] yellow or green. The first set on the right is colored red/pink, the second set is colored dark red, and the third set is colored blue.

- $$P_s = \phi_s(P_t) \cup \phi_s(Q_t) \cup R_D \circ \phi_s(Q_t).$$

  P_s is the union of all the pieces not colored blue. The first set on the right is colored pink/red/green, and the second set is colored dark red, and the third set is colored yellow.

- $Q_s = \phi_s(Q_t)$. This set is colored dark red.

We can interpret all these equations as the desired patch coverings. ♠

[3]The grey tile, which is a central tile, is always excluded.

Lemma 4.10 *Lemma 4.6 is true when $s < 1/2$ and $K(s) > 1$.*

Proof: Figure 4.11 shows a representative picture for s in this range.

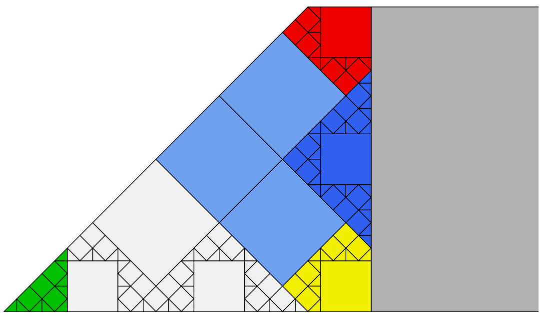

Figure 4.11: The tiling Δ_s^0 for $s = 12/29$. Here $K(s) = 1$.

- $A_s = R_D \circ \phi_s(A_t) \ \cup \ R_H \circ R_D \circ \phi_s(A_t) \ \cup \ R_D(\nu') \ \cup \ R_H \circ R_D(\nu') \cup \Theta,$

$$\nu' = \bigcup_{k=1}^{K-1} \Psi_s^k.$$

  Here Θ is a finite union of square tiles, colored light blue. The set A_s is the union of tiles in X_s^0 not colored white or green (or grey). The first set on the right is colored red. The second set is colored yellow. The union of the third and fourth sets is colored dark blue.

- $$B_s = \nu \cup R_V(\nu) - \Theta, \qquad \nu = \bigcup_{k=0}^{K-1} \Psi_s^k.$$

  Here Θ is a finite union of squares (colored light blue) which we delete from our union. The point is that the top square of each Ψ_s^k lies above B_s. B_s is colored white and green.

- $P_s = A_s \cup R_D(A_s)$.

- $Q_s = \phi_s(B_t)$.

We can interpret all these equations as the desired patch coverings. ♠

Lemma 4.11 *Lemma 4.6 is true when $s < 1/2$.*

Proof: In view of what we have already shown, it suffices to consider the case when $K(s) = 1$. Figure 4.12 shows a representative picture in this case.

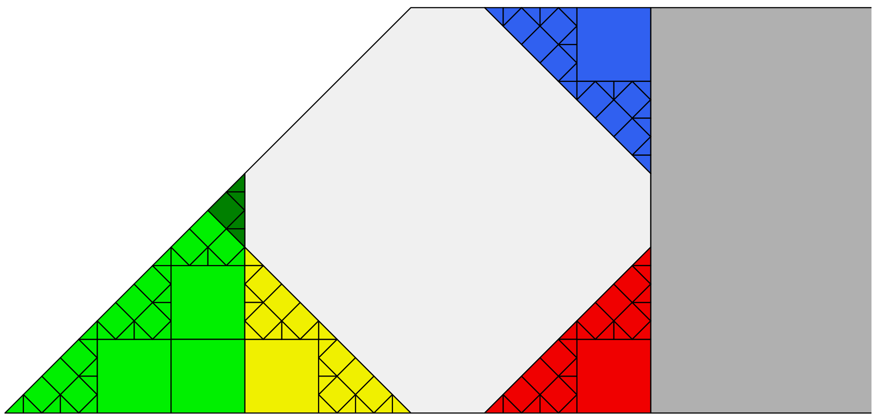

Figure 4.12: The tiling Δ_s^0 for $s = 11/35$. Here $K(s) = 1$.

- $$A_s = R_D \circ \phi_s(A_t) \cup R_H \circ R_D \circ \phi_s(A_t) \cup \phi_s(Q_t) \cup \Theta.$$
  A_s is colored red/white/blue/green. $\phi_s(A_t)$ is colored yellow. The sets on the right are respectively colored blue, red, green, and white.
- $B_s = \phi_s(P_t)$. This set is colored yellow and light green.
- $$P_s = \phi_s(A_t) \cup R_D \circ \phi_s(A_t) \cup R_H \circ R_D \circ \phi_s(A_t) \cup \Theta.$$
  Here P_s is the union of all tiles not colored green (or grey). The sets on the right are, respectively, colored yellow, blue, red, and white.
- $Q_s = \phi_s(B_t)$. This set is colored light and dark green.

We can interpret all these equations as the desired patch coverings. ♠

We have exhausted all the cases. This completes the proof of Lemma 4.6.

5 Proof of Statement 1

5.1 The Easy Direction

Our goal is to prove that $\widehat{\Lambda}_s$ is a disjoint union of two arcs if and only if s is oddly even. Let S_s denote the left half of $\widehat{\Lambda}_s$. By symmetry, the result we want is equivalent to the statement that S_s is an arc if and only if s is oddly even.

First we prove that S_s is an arc only if s is oddly even. This is the easier of the two directions.

Lemma 5.1 *For each integer k such that $R^k(s) > 1/2$, there is an octagon O_k which having one edge in the left side of X_s and one edge in the bottom side of X_s. If there are two istinct indices k and ℓ with this property, then the octagons O_k and O_ℓ are distinct.*

Proof: Say that an octagon is *wedged* in a parallelogram (of the kind we are considering) if one edge of the octagon lies in the left edge of the parallelogram and another edge lies in the bottom edge.

Let $s_0 = s$ and $s_k = R^k(s)$. For ease of notation, we set $\phi_k = \phi_{s_k}$ and $\Delta_k = \Delta_{s_k}$, etc. When $s_k > 1/2$, the central tile C_k of Δ_k is an octagon wedged into X_k. By Theorem 2.5, the octagon $\phi_{k-1}(C_k)$ is a tile of Δ^0_{k-1}, and is wedged into X_{k-1}.

Iterating Theorem 2.5, we see that

$$O_k = \phi_0 \circ ... \circ \phi_{k-1}(C_k) \tag{21}$$

is wedged into X_0.

Suppose that $\ell > k$ is another index such that $s_\ell > 1/2$. Then the two octagons

$$C_k, \qquad \phi_k \circ ... \circ \phi_{\ell-1}(C_\ell)$$

are distinct because one octagon is the central tile of Δ_k and the other one is not. But then O_k and O_ℓ are the images of the above octagons under the same similarity. Hence, they are distinct. ♠

Corollary 5.2 *If $R^n(s) > 1/2$ for at least K different positive indices, then S_s has at least $K+1$ connected components.*

Proof: The K octagons guaranteed by Lemma 5.1 are all distinct. Call these octagons $O_1, ..., O_K$. Each of these octagons is wedged into X_s, and so the union of these octagons separates X_s^0 into $K+1$ connected components. We just need to see that S_s intersects each component.

Each octagon O_j has two vertices in the bottom edge of X_s. At each of these vertices, the adjacent edge of O_j makes an acute angle with the bottom edge of X_s. (The angle is $\pi/4$.) Since the Δ_s consists of an open dense (in fact full measure) set of squares and semi-regular octagons, every neighborhood of the two vertices in question must intersect infinitely many tiles of Δ_s. Hence, the two bottom vertices of O_j lie in S_s. This proves what we want. ♠

What we have shown is that S_s is not an arc if $R^n(s) > 1/2$ for some $n > 0$.

Lemma 5.3 *If $s > 1/2$, then S_s is not an arc.*

Proof: Let C be the trivial tile of Δ_s. The same argument as in the previous lemma shows that S_s contains the following three points.

1. The vertex where the vertical edge of C meets the left edge of X_s. This is the top vertex of both A_s and P_s.

2. The vertex where the slope -1 edge of C meets the bottom edge of X_s. This is the right vertex of B_s and of Q_s.

3. The bottom left vertex of X_s. Thie is the left vertex of B_s and of P_s.

Any arc connecting these 3 points, in some order, must cross at least twice one of the symmetry lines from the Pinching Lemma. Hence S_s cannot be an arc. ♠

Remark: In fact S_s is homeomorphic to a "Y" when $s > 0$ and $R^n(s) < 1/2$ for all $n > 0$. However, we do not prove this.

We now know that S_s is an arc only if s is oddly even. The rest of the chapter is devoted to proving that S_s is an arc when s is oddly even.

5.2 A Criterion for Arcs

Say that a *marked piece* is an compact, embedded, convex set with two distinguished vertices. Say that a *chain* is a finite union $D_1, ..., D_n$ of marked pieces such that $D_i \cap D_{i+1}$ is one point, and that this point is one of the marked points on each of D_i and D_{i+1}. We also require that $D_i \cap D_j = \emptyset$ for all other indices $i \neq j$. We define the *mesh* of the chain to be the maximum diameter of one of the marked pieces.

We say that a compact set S *fills* a chain $D_1, ..., D_n$ if $S \subset \bigcup D_i$ and S contains every marked point of the chain. The purpose of this section is to establish the following (certainly well known) criterion.

Lemma 5.4 (Arc Criterion) *Let S be a compact set. Suppose, for every $\epsilon > 0$, that S fills a chain having mesh less than ϵ. Then S is an embedded arc.*

We will assume that S satisfies the hypotheses of the lemma, and then show that S is an arc. First of all, S is clearly connected.

Lemma 5.5 *Suppose that S fills a chain $C_1, ..., C_m$. Then there is some $\epsilon > 0$ with the following property. If S also fills a chain $D_1, ..., D_n$ having mesh size less than ϵ, then S fills a chain $E_1, ..., E_p$ where each E_i has the form $C_j \cap D_k$.*

Proof: We can choose ϵ so small that each D_j has following properties.

- The diameter of D_j is smaller than the length of any edge of any C_i.
- D_j intersects any C_i in at most 2 edges.
- D_j cannot intersect C_i and C_k if i and k are not consecutive indices.

For each i, there are unique and distinct pieces D_{j_1} and D_{j_2} which contain the two marked points of C_i. We claim that the pieces between D_{j_1} and D_{j_2} must have both marked points inside C_i. Assuming that this is true, we form the portion of the E-chain insid C_i by taking the intersections $C_i \cap D_j$ and using the marked points of D_j for $j_1 < j < j_2$. The marked points of $C_i \cap D_{j_1}$ are the marked point of C_i inside of D_{j_1} and the marked point of D_{j_1} inside C_i. Similarly for $C_i \cap D_{j_2}$. We do the same thing for each i and this gives us the conclusion of the Lemma.

Now we establish our claim. If our claim was false, then some D_j, with $j_1 < j < j_2$, would have one marked point in C_i. Note that D_j must have another marked point in either C_{i-1} or C_{i+1}, because this marked point is a vertex of S. Suppose that D_j has its other marked point in C_{i+1}. Then $D_j \cap (C_i \cup C_{i+1})$ is disconnected because D_j does not contain the vertex $C_i \cap C_{i+1}$. Hence

$$D_1 \cup ... \cup D_{j-1} \cup (D_j \cap (C_i \cup C_{i+1})) \cup D_{j+1} \cup ... \cup D_n$$

consists of two disconnected components, each of which intersects S nontrivially. This contradicts the connectivity of S. ♠

If the chain $C_1, ..., C_m$ and the chain $E_1, ..., E_p$ are related as in the previous lemma, we say that $E_1, ..., E_p$ *refines* $C_1, ..., C_m$. In view of the previous result, we can assume that S fills an infinite sequence $\{\Omega_i\}$ of chains such that each one refines the previous one and the mesh size tends to 0.

For each i, we inductively create a partition P_i of $[0, 1]$ into intervals, such that the number of intervals coincides with the number of marked pieces in Ω_i, in the following manner. Once P_i is created, we distribute the intervals of P_{i+1} according to how Ω_i contains Ω_{i+1}. If the kth piece of Ω_i contains n_k pieces of Ω_{i+1}, then P_{i+1} is created from P_i by subdividing the kth interval of P_i into n_k intervals of equal size. Note that the mesh size of P_i tends to 0 as i tends to ∞.

There is a bijective correspondence between marked pieces in the chains and intervals in the partition. The correspondence respects the containment and intersection properties. For instance, two marked pieces intersect if and only if the corresponding intervals share an endpoint. Each point of S is contained in an infinite nested intersection of marked pieces, and we map this point to the corresponding nested intersection of intervals. This map is clearly a homeomorphism. The inverse map gives a parameterization of S as an arc in the plane.

5.3 Elementary Properties of the Limit Set

Let A_s be the set from §2.7.

Lemma 5.6 *S_s contains the two left vertices of X_s and the two obtuse vertices of A_s.*

Proof: Let v be one of left vertices of X_s. Since the angle of X_s at v is not a right angle, there must be infinitely many squares contained in every neighborhood of v.

Note that the top left vertex of X_s^0 is also the top obtuse vertex of A_s. So, S_s contains the top obtuse vertex of A_s. By symmetry, S_s contains the bottom obtuse vertex of A_s. ♠

By Lemma 2.7, Δ_s consists entirely of squares. As we remarked after proving Lemma 2.7, these squares are either boxes or diamonds.

Lemma 5.7 *Suppose $\gamma \subset X_s^0$ is a compact arc which connects a point in a box to a point in a diamond. Then γ contains a point of S_s.*

Proof: Compare [**S0**, §8]. By compactness, it suffices to show that arbitrarily small perturbations of γ contain points of S_s. Hence, we may perturb so that γ does not contain any vertices of any tiles in Δ_s. Suppose γ does not intersect $\widehat{\Lambda}$. Then γ only intersects finitely many tiles, $\tau_1, ..., \tau_n$. Moreover, τ_i and τ_{i+1} must share an edge. Hence, by induction, τ_1 is a box if and only if τ_n is a box. But τ_1 is a box and τ_n is a diamond. This is a contradiction. ♠

Lemma 5.8 *Each fundamental line of symmetry contains a point of S_s.*

Proof: To make the argument cleaner, we attach a large diamond δ to the picture along the left edge of X_s, and we attach a large box β to the picture along the bottom edge of X_s. These extra squares are disjoint from X_s except along the relevant edges. Once we add these two squares, we see that each of the lines in question connects a diamond to a box. H connects δ to a the leftmost central tile of Δ_s and both V, D, E all connect β to δ. By Lemma 5.7, each of these lines contains a point of S_s. ♠

5.4 The End of the Proof

Suppose that S_s fills some chain $D_1, ..., D_n$. We call this chain *good*

- D_j is disjoint from the interiors of the edges of ∂A_s, for $j = 1, ..., n$.
- The first marked point of D_1 is the bottom left vertex of X_s.

- The last marked point of D_n is the top left vertex of X_s.

Lemma 5.9 *S_s fills a good chain.*

Proof: Our chain has two pieces. We set $D_1 = B_s$, the triangle from §2.7, and we let D_2 be the set from Corollary 4.3. ♠

Lemma 5.10 *Let $t = R(s)$. Suppose S_t fills a good chain having mesh m. Then S_s fills a good chain having mesh at most $m/\sqrt{2}$.*

Proof: Let Ω_t be the good chain filled by S_t. We use the notation from the Filling Lemma, Equation 8, and Theorem 2.5. We make our construction in 5 steps.

Step 1: For $j = 0, ..., K$, we define

$$\Omega_j = T_s^j \circ \phi_s(\Omega_t). \qquad (22)$$

Figure 5.1 illustrates our construction. The individual chains $\Omega_0, ..., \Omega_{K-1}$ piece together to make one long chain because the second disk of Ω_j touches the common edge between Ψ_s^j and Ψ_s^{j+1} only at the bottom vertex of this edge.

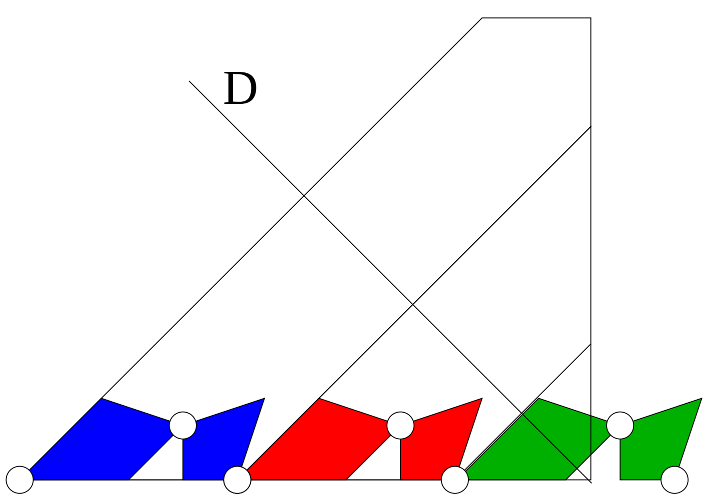

Figure 5.1: Step 1: The chains Ω_j for $j = 0, ..., K$. Here $K = 2$.

Step 2: The problem with Ω_K is that some of it sticks over the edge of X_s^0. This is the green set in Figure 5.1. However, we know from the Pinching Lemma that S_s intersects the line D_s in a single point. All other points of D_s must have neighborhoods contained in finitely many squares. For this reason, we can make the essentially the same construction as in Corollary 4.3 to produce a convex disk $U \subset \Psi_s^K$ such that $S_s \cap \Psi_s^K \subset U$ and $U \cap D_s$ is the single point which belongs to S_s.

We now improve Ω_K as follows. We intersect each piece of Ω_K with the set U and throw out all those after the first one which is disjoint from U.

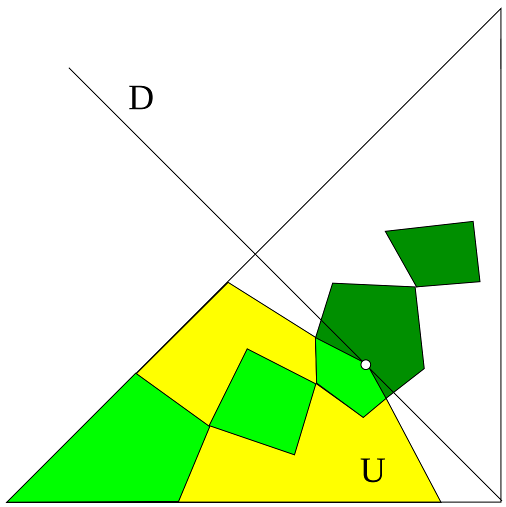

Figure 5.2: Step 2: Improving Ω_K.

The result is a chain which joins the bottom vertex of the edge $\Psi_s^{K-1} \cap \Psi_s^k$ to the point $S_s \cap D_s$. Figure 5.2 shows the construction. We call this improved chain Ω'_K.

Define

$$\Upsilon_0 = \Omega_0, ..., \Omega_{K-1}, \Omega'_K. \tag{23}$$

By construction, this chain is filled by the portion of S_s beneath the line D_s.

Step 3: Define

$$\Upsilon_2 = \Upsilon_0, \Upsilon_1; \qquad \Upsilon_1 = R_D(\Upsilon_0) \tag{24}$$

That is, we continue our chain by reflecting it across the line D. The resulting chain contains S_s, by symmetry, but we are not quite done.

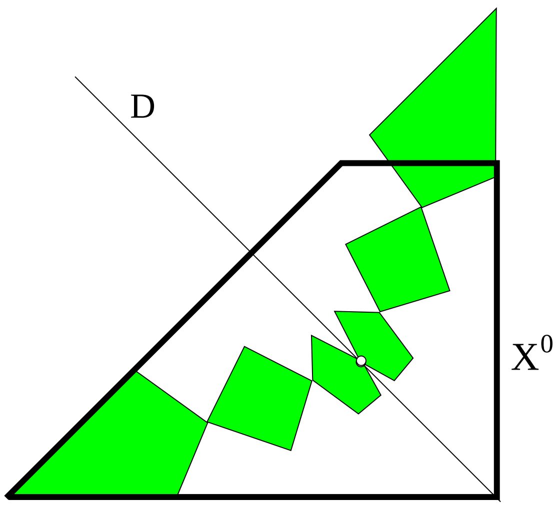

Figure 5.3: Step 3: Extending by reflection

Some of the final pieces of Υ_1 might not lie in X_s. The problem is that X_s is not symmetric with respect to R_D. The portion below D is larger than the portion above D.

Step 4: We finish the construction by a method very similar to what we did in Step 2. We simply intersect all the pieces of Υ_1 with X_s^0, and let Υ_1' and omit all those pieces which come after the first one which has trivial intersection with X_s^0. We set $\Upsilon_3 = \Upsilon_0, \Upsilon_1'$. By construction, Υ_3 is a chain filled by S_s. Moreover, since ϕ_s contracts distances by some $\lambda_s < 1/\sqrt{2}$. we see that the mesh of Υ_3 is less than $m/\sqrt{2}$.

Step 5: The chain Υ_3 might not be clean. To remedy this, we shrink the pieces slightly (away from the marked points) so that they are all disjoint from the interiors of the edges of A_s. What allows us to do this is the Shield Lemma combined with compactness. The final chain has all the desired properties. ♠

Note that if s is oddly even, then so is $R(s)$. The chains in Lemma 5.9 all have mesh size less than 2. It now follows from iterating Lemma 5.10 that S_s fills a good chain having mesh size less than ϵ, for any given ϵ_0. Our Arc Criterion how shows that S_s is an arc. This completes the proof of Statement 1 of the Main Theorem.

6 Proof of Statement 2

6.1 The Main Argument

In this chapter we suppose throughout that $s \in (0,1)$ is irrational and $R^n(s) > 1/2$ only finitely often. This means that there is some n such that $R^n(s)$ is oddly even. Let $f(s)$ denote the smallest n with this property. Let S_s denote the left half of $\widehat{\Lambda}_s$, as in the previous chapter. Our goal is to show that S_s is a finite forest.

Lemma 6.1 *Both $S_s \sqcap A_s$ and $S_s \sqcap B_s$ are finite unions of arcs.*

Proof: When $f(s) = 0$, the result is true by Statement 1 of the Main Theorem. Suppose by induction that this lemma is true for all s such that $f(s) < N$. If s is chosen so that $f(s) = N$, then let $t = R(s)$. Then $f(t) = N - 1$. By induction, both $S_t \sqcap A_t$ and $S_t \sqcap B_t$ are finite unions of arcs. By the Filling Lemma, both $\widehat{\Lambda}_s \sqcap A_s$ and $\widehat{\Lambda}_s \sqcap B_s$ are contained in finite unions of similar copies of $\widehat{\Lambda}_t \sqcap A_t$ and $\widehat{\Lambda}_t \sqcap B_t$. The argument is very similar to what we did to show that S_s is connected in the oddly even case. Hence, both both $S_s \sqcap A_s$ and $S_s \sqcap B_s$ are finite unions of arcs.

Corollary 6.2 *S_s is a finite union of arcs.*

Proof: We have $S_s = (S_s \sqcap A_s) \cup (S_s \sqcap B_s)$. ♠

Following this section, the entire chapter is devoted to proving the following result.

Theorem 6.3 (No Loops) *Let $s \in (0,1)$ be any irrational number. Then S_s contains no embedded loops.*

But a finite union of arcs which contains no embedded loops must be a finite forest. Modulo the No Loops Theorem, this completes of Statement 2 of the Main Theorem.

We prove the No Loops Theorem using the same kind of extremality argument we gave for the Pinching Lemma. One case of this argument is much harder than the others, and so we split it off and tackle it first. Once we build the machinery for this one case, the rest of the proof is easy.

6.2 Interaction with the Octagrid

Suppose that $s < 1/2$ and $t = R(s) < 1/2$. We consider the octagrid components defined in §3.3. Here is the main result of this section.

Lemma 6.4 *An embedded loop in S_s lies in a single octagrid component.*

We will prove this through a series of smaller results. Say that an *octagrid edge* is an edge of one of the octagrid components. Let $u = R(t)$.

Lemma 6.5 *S_s intersects each octagrid segment in at most one point provided that $u < 1/2$.*

Proof: Let G be an octagrid component and let σ be an edge of G. Just as in the proof of Lemma 3.3, we can use bilateral symmetry and the map T_s from Equation 8 to reduce to the case where $G \subset Z_s^0$. If one endpoint of σ is contained in a square of the pyramid associated to Δ_s, then G is entirely contained in that square. This case is trivial. So, σ must be one of the 8 segments emanating from the center of the bottom left tile of the extended pyramid associated to Δ_s.

By Theorem 2.5, it suffices to prove that $\sigma' = \phi_s^{-1}(\sigma)$ intersects at most one point of S_t, where $t = R(s)$. One endpoint of σ' is the center c' of the central tile of Z_t. We think of σ' as pointing away from c'. Since $u < 1/2$, the central tile of Z_t is a square. The 4 nontrivial cases are as follows.

1. σ' is horizontal and points left.
2. σ' has slope 1 and points down.
3. σ' is vertical and points down.
4. σ' has slope 1 and points up.

In the other 4 cases, σ' is entirely contained in the central tile of Z_t. In the 4 nontrivial cases, σ' is longer than an octagrid edge associated to the parameter t, but this does not bother us.

The line D_t of symmetry contains the point c'. Reflection in D_t reduces Case 3 to Case 1 and Case 4 to Case 2. So, it suffices to consider Cases 1 and 2. In Case 1, σ' is a segment of H^0. In this case, we apply the Pinching Lemma. In Case 2, we apply Theorem 2.5 again: $\sigma'' = \phi_t^{-1}(\sigma')$ is the left half of H and intersects S_u at most once by the Pinching Lemma. ♠

Lemma 6.6 *S_s intersects each octagrid segment in at most one point provided that $u > 1/2$.*

Proof: The proof here is the same, except that the central tile of Z_t is an octagon. In this case, we immediately renormalize so that $\sigma'' = \phi_t^{-1}(\sigma')$ is one of the 8 segments emanating from the center $(0, 0)$ of the trivial tile of Δ_u. After reflecting in the origin, we arrive at 2 nontrivial cases.

1. σ'' is horizontal and points left.

2. σ'' has slope 1 and points downward.

In the first case σ'' is simply the portion of H lying to the left of the origin. This case follows from the Pinching Lemma applied to the parameter u. In the second case, σ'' agrees with the line E_u of symmetry outside the trivial tile of Δ_u. Again, this follows from the Pinching Lemma applied to the parameter u. ♠

We record the obvious corollary.

Corollary 6.7 *S_s intersects each octagrid segment in at most one point.*

Suppose now that γ is an embedded loop in S_s.

Lemma 6.8 *γ cannot link any vertex of an octagrid component.*

Proof: γ cannot surround any square in the pyramid associated to s, because this pyramid is a topological disk sharing an arc with ∂X_s.

Say that a *peripheral tile* is a tile of the extended pyramid which is not a tile of the pyramid. The peripheral tiles are colored blue in Figures 5.8 and 5.9.

Say that a *blocker* is an octagrid edge which lies entirely in a tile and has a vertex in ∂X_s^0. These segments are disjoint from S_s and hence from γ. Each peripheral tile has at least one blocker. But then γ cannot surround a peripheral tile because it would have to intersect a blocker.

The remaining vertices of the octagrid (meaning, the vertices of the octagrid components) have the following structure. At least one octagrid edge emanating from the vertex lies entirely inside a tile. For this reason, γ cannot link any vertex of an octagrid component. ♠

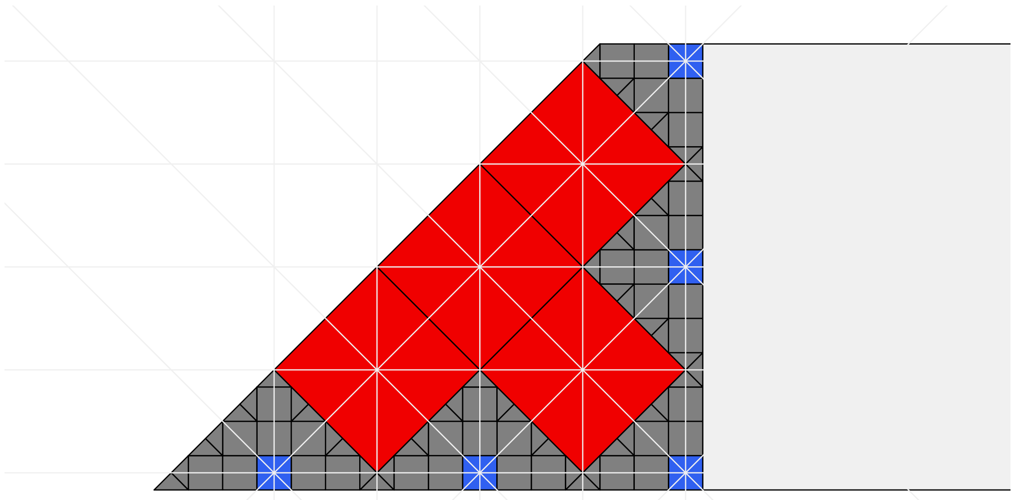

Figure 6.1: The octagrid for $s = 13/32$.

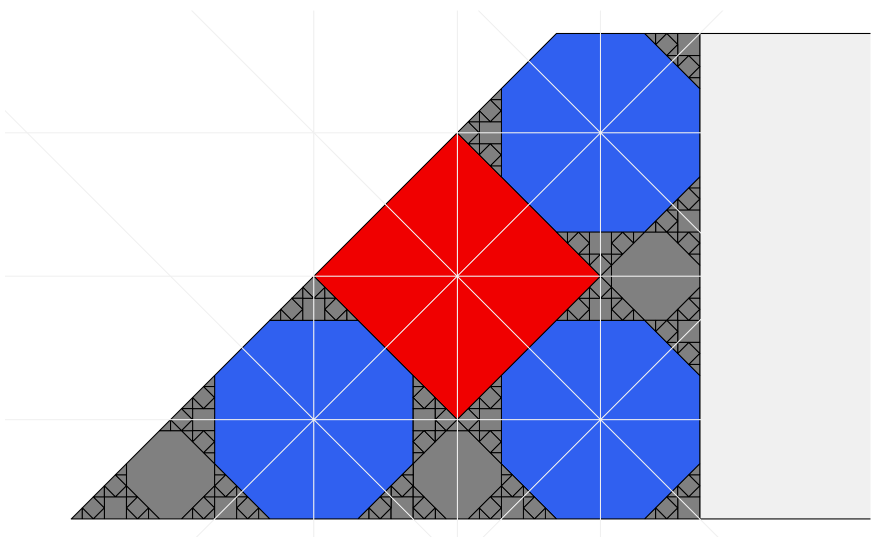

Figure 6.2: The octagrid for $s = 22/57$.

If γ is not contained in a single octagrid component, and γ does not link any of the vertices of octagrid components, then γ must cross some octagrid edge twice. We have already ruled this out. Hence γ is contained in a single octagrid component. This completes the proof of Lemma 6.5.

6.3 No Embedded Loops

Now we prove the No Loops Theorem. Our method of proof is very much like what we did for the Pinching Lemma. Say that a *counterexample* is a pair $\Omega = (\gamma, s)$ where γ is an embedded loop in S_s. We define

$$\lambda(\Omega) = \frac{\mathrm{diam}(\gamma)}{g(s)}, \qquad g(s) = \mathrm{diam}(X_s^0). \tag{25}$$

Lemma 6.9 *For any $\epsilon > 0$ there exists a counterexample Ω, whose parameter is less than $1/2$, such that $\lambda(\Omega) > \lambda - \epsilon$.*

Proof: As in the proof of Lemma 4.5, it suffices to consider the case of a counterexample Ω having $s \in (1/2, \sqrt{2}/2]$. In this case, the reflection R_V maps the region to the right of V into Z_s^0 and the reflection R_D maps the region above D_s into Z_s^0. Figure 6.3 shows a fairly typical example.

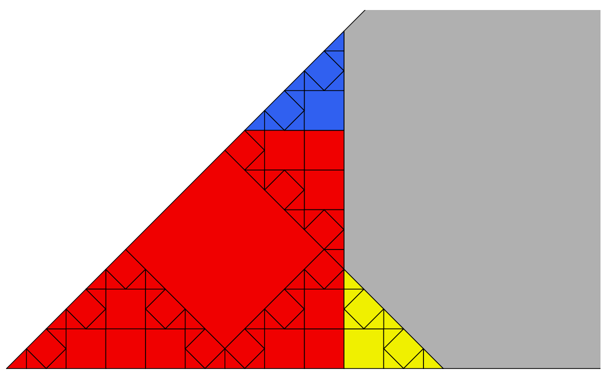

Figure 6.3: Z_s^0 (red) and the components of $X_s^0 - Z_s$ for $s = 11/17$.

By the Pinching Lemma, γ can intersect each of V and D_s at most once. Hence γ lies to one side or the other of each of these lines. So, by symmetry, we can assume that $\gamma \subset Z_s^0$. The rest of the proof is as in Lemma 4.5. ♠

As in the proof of the Pinching Lemma, we now analyze potential counterexamples having parameter $s \in (1/4, 1/2)$. First suppose $s \in (1/4, 1/3)$. For s in this range, we have $t = R(s) > 1/2$. Let O_s be the image, under the ϕ_s^0 of the central tile of Δ_s. This octagon separates X_s^0 into three regions having disjoint closures, as shown in Figure 6.3. One of the regions, colored red in Figure 6.3, is precisely Z_s^0.

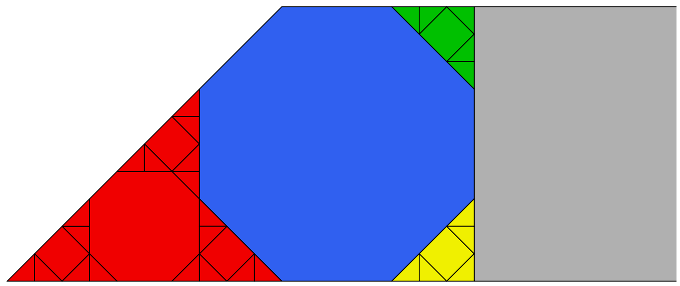

Figure 6.4: Z_s^0 (red) and other regions for $s = 5/17$.

The reflections R_D and R_V respectively carry the other regions into the one contined in Z_s. Hence, we may assume by symmetry that $\gamma \subset Z_s$. We now get the same contradiction as we got in the proof of the Pinching Lemma.

For $s \in (1/3, 1/2)$ we have $R(s) < 1/2$. Lemma 6.5 applies, and so we know that our counterexample is contained in a single octagrid component. But then, by Lemma 3.3, we can assume by symmetry that our counterexample lies in Z_s. This gives us the same contradiction as in the Pinching Lemma.

This completes the proof that $\widehat{\Lambda}_s$ has no embedded loops.

7 Proof of Statement 3

7.1 The Limit Set is Perfect

Recall that a closed set C is *perfect* if every point $p \in C$ is an accumulation point of $C - \{p\}$.

Lemma 7.1 *When s is irrational, $\widehat{\Lambda}_s$ is a perfect set.*

Proof: If $\widehat{\Lambda}_s$ is not perfect, then there is some $p \in \widehat{\Lambda}_s$ and come open disk U containing p such that $U \cap \widehat{\Lambda}_s = p$. The open set U must contain infinitely many tiles of Δ_s, because $\widehat{\Lambda}_s \cap U$ is nonempty. Therefore, if we write Δ_s as in the Covering Lemma, the image of some patch must have p as an accumulation point. Choosing ϵ small enough, we can guarantee that there exists an ϵ-patch (K, ψ, ϵ) such that $\psi(K) \subset U$.

If K is a triangle, then two of the vertices v_1 and v_2 of K have acute angles. (The angle is $\pi/4$.) These vertices must be accumulation points of infinitely many tiles, because all the tiles are squares and semi-regular octagons. But then $\psi(v_1)$ and $\psi(v_2)$ are accumulation points of infinitely many squares in Δ_s. Hence $\widehat{\Lambda}_s \cap U$ contains at least 2 points. This is a contradiction.

If K is a pentagon, then K has 2 vertices v_1 and v_2 with obtuse angles. (The angle is $3\pi/4$.) If v_1 is not an accumulation point of infinitely many tiles of $\Delta_u \cap K$, then v_1 is the vertex of some octagon of $\Delta_u \cap K$, But then there are two new acute vertices w_1 and w_2 which must be accumulation points of infintely many tiles of $\Delta_u \cap K$. This gives us the same contradiction as above. The only way out of the contradiction is for both v_1 and v_2 to be accumulation points of infinitely many tiles of $\Delta_u \cap K$, but this is again a contradiction. There is no way out. ♠

7.2 Overview for the Rest of the Proof

We call $s \in (0, 1)$ an *octagonal parameter* if $R^n(s) > 1/2$ infinitely often. The reason for the name is that, thanks to Theorem 1.1, Δ_s contains infinitely many octagons if and only if s is octagonal. Our remaining goal is to prove that $\widehat{\Lambda}_s$ is a Cantor set when s is octagonal. Let S_s denote the left half of $\widehat{\Lambda}_s$, as usual.

We already know that S_s is closed and perfect. It remains to show that S_s is totally disconnected for any octagonal s. The proof is a bootstrap argument. The basic idea is that renormalization tends to make connected components of S_s larger. So, we will start with the assumption that S_s has a nontrivial connected component when s is octagonal, and ultimately we will produce a new octagonal parameter u for which S_u has a very large and egregious kind of connected component, which we can rule out. Throughout the proof, K_s will stand for a *symmetric piece*, one of the sets A_s, B_s, P_s, Q_s from §2.7.

7.3 Unlikely Sets

Lemma 7.2 *Let $s \in (0,1)$ be irrational. Let K_s be a symmetric piece. Let e be any edge of K_s. Then some tile of $\Delta_s \cap K_s$ has an edge in e.*

Proof: We do this in a case-by-case way.

1. Suppose $s < 1/2$ and $K_s \in \{B_s, Q_s\}$. Applying Theorem 2.5 repeatedly, we see that there are infinitely many tiles which have edges in the bottom edge of X_s and infinitely many tiles which have edges in the left edge of X_s. Eventually these tiles lie in both B_s and Q_s. This takes care of two out of three edges of B_s and Q_s. The third edge, in each case, follows from bilateral symmetry.

2. Suppose $s < 1/2$ and $R(s) > 1/2$ and $K_s \in \{A_s, P_s\}$. In this case, there is an octagon having the same width as the central tiles, and this octagon has edges in all the sides of A_s. Likewise, the same octagon has edges in all the sides of P_s. See Figure 4.12.

3. Suppose $s < 1/2$ and $R(s) < 1/2$ and $K_s \in \{A_s, P_s\}$. In this case, each edge of A_s has a segment which is an edge of one of the tiles in the extended pyramid. The same goes for P_s. See Figures 3.8 and 3.9.

4. Suppose $s > 1/2$. The argument given in Case 1 takes care of P_s and B_s. The case of A_s and Q_s follows from inversion symmetry. See Remark (i) in §2.7.

Thus exhausts the possibilities. ♠

Let C be a nontrivial connected subset of $\widehat{\Lambda}_s$. Let $\mathcal{K}$ be a patch cover, as in the Covering Lemma. What we mean is that $\mathcal{K}$ is a finite union of patches and tiles, as in Equation 20. We call C *bad* with respect to $\mathcal{K}$ if C does not intersect the interiors of the images of any of the patches. That is, C is disjoint from the interiors of all the sets $\psi_j(K_j)$.

Call C *unlikely* if, for every $\epsilon > 0$, there is a patch covering $\mathcal{K}$ of scale less than ϵ, so that C is bad with respect to $\mathcal{K}$. So, C is bad with respect to an infinite sequence of patch covers, having scale tending to 0.

Lemma 7.3 *Let $s \in (0,1)$ be irrational. There do not exist any unlikely subsets of $\widehat{\Lambda}_s$.*

Proof: We will suppose some unlikely set C exists and get a contradiction. By taking a suitable subset of C, we can assume that C is a line segment contained in the boundary of some particular patch of one of the sequence of bad covers. By Lemma 7.2, some segment of C lies in a tile boundary. Further shrinking C, we can assume that C is one edge of some tile τ_1.

The midpoint m of C is the accumulation point of infinitely many tiles of Δ_s. These tiles are all disjoint from τ_s. Hence, there is some patch (ψ, K, u) so that $m \in \psi(K)$. But C cannot intersect the interior of $\psi(K)$. Hence, one edge of $\psi(K)$ lies in the line containing C. Shrinking the scale as needed, we can assume that one edge of $\psi(K)$ is contained in C.

Some tile τ_2 of $\psi(K) \cap \Delta_s$ has an edge in C, by Lemma 7.2. But then some point of C lie on the common boundary of τ_1 and τ_2 and cannot belong to the limit set. This is a contradiction. ♠

7.4 Tails

For each symmetric set $K_u \in \{A_u, B_u, P_u, Q_u\}$, and each subset $C \subset S_u$, let $C \sqcap K_u$ denote the set of points $p \in C$ such that every neighborhood of p contains infinitely many tiles of $K_u \cap \Delta_u$. Note that $C \sqcap K_u$ might be a proper subset of $C \cap K_u$, but the two sets agree on the interior of K_u. The former set is easier to pull back using the patches from the Covering Lemma.

We say that the symmetric piece K_u has a *tail* if some connected component of $S_u \sqcap K_u$ contains both an interior point of K_u and a boundary point of K_u. Conceptually, we think of a tail as a little arc which joins a boundary point of K_u to an interior point, but we don't actually know that our connected set is path connected.

Lemma 7.4 *Suppose S_s is not totally disconnected. Then for all sufficiently large n the parameter $u = R^n(s)$ has the following property. At least one of the 4 symmetric pieces K_u has a tail.*

Proof: If S_s is not totally disconnected, then it has a closed connected subset C having more than one point. By Lemma 7.3, once n is large enough and $u = R^n(s)$, we can find a patch (K, ψ, u) such that some point $p \in C$ lies in the interior of $K' = \psi(K)$ and some point of C lies outside of K'.

Consider the set $C \sqcap K'$. For the sake of exposition, assume first that C is path connected. Then we can find some path $\alpha \in C$ joining p to some point of C lying outside K'. Let β be the maximal initial portion of α which lies in K'. By construction $\beta \subset C \sqcap K'$ and β joins p to a point $q \in \partial K'$. By definition of a patch, $\psi^{-1}(\beta)$ is a path in $S_u \sqcap K_u$ joining $\psi^{-1}(p)$ to $\psi^{-1}(q)$. The former point lies in the interior of K and the latter point lies on the boundary. This gives K_u a tail.

We follow the same outline when C is merely connected. Define an ϵ-*chain* to be a sequence of points $p_0, ..., p_N \in C$ such that $\|p_k - p_{k+1}\| < \epsilon$ for all k. We say that this chain joins p_0 to p_N. Any open neighborhood of a connected set is path connected. Hence, for any m, there is a $(1/m)$-chain α_m joining p to some point of C outside of K'. Let β_m be the maximal initial portion of α_m which lies entirely in K'. Passing to a subsequence, we can assume that the endpoints of β_n converge. One of the endpoints is always p. Let q be the limit of the other endpoints. By construction $q \in \partial K'$.

Let β be the connected component of $C \sqcap K'$ containing p. By construction $q \in C \sqcap K'$. Moreover, for every $\epsilon > 0$ there is an ϵ-chain connecting p to q. Hence $q \in \beta$. Now we pull back by ψ to give K_u a tail, as in the path connected case. ♠

The Chain Trick: We will have many occasions below to make the same kind of argument as we just gave, and the geometry behind the argument is always clearer in the path connected case. For this reason, we will give the arguments in the path connected case and then remark that the same trick as used above – i.e., using an infinite sequence of chains in place of a path – handles the general case. For reference, we call this the *Chain trick:*

7.5 Crosscuts

Let K be a (solid) triangle and let $\Lambda \subset K$ be a closed set. We say that a *crosscut* for (K, Λ) is a connected subset of Λ which contains two points $p, q \in \partial K$. We require that p and q are not both contained in the interior of the same edge of K.

In the following result, it is not really essential that s is octagonal and that $s, u > 1/2$. However, in this case, the symmetric pieces associated to the two parameters are all right-angled isosceles triangles. This makes the geometry easier and cuts down on the number of cases to consider.

Lemma 7.5 *Suppose that $s > 1/2$ is octagonal and S_s is not totally disconnected. Then for all sufficiently large n the parameter $u = R^n(s)$ has the following property provided that $u > 1/2$: The pair $(K_u, K_u \sqcap S_u)$ has a crosscut for some symmetric piece K_u.*

Proof: By Lemma 7.4, we can assume without loss of generality that some K_s has a tail C. As remarked above, we will assume that C is a path; the Chain Trick then finishes the proof.

Let $\epsilon = \|p - q\|$ and choose n so large that the patch covering in the Covering Lemma corresponding to $u = R^n(s)$ has scale much smaller than ϵ. There is a patch (K_j, ψ_j, u) such that $K' = \psi_j(K_j)$ contains p. There are several cases to consider, as shown in Figure 7.1. (The third part of Figure 7.1 just shows one of the several possibilities for that case.)

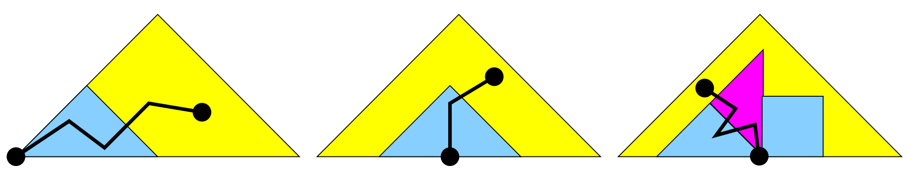

Figure 7.1: Possibilities for the patch covering.

- Suppose that p is a vertex of K_s. Then p is also a vertex of $K' = \psi_j(K_j)$. The two sides of K' incident to p lie in sides of K. But then the path C must exit through the third side of K' in order to reach q. The pullback $\psi_j^{-1}(C)$ is a crosscut of K_j.

- Suppose p lies in the interior of an edge of K and also in the interior of an edge of K'. Then, again, C must exit K' through one of the other edges of K'. Now we pull back as before.

- Suppose p lies in the interior of an edge of K and is a vertex of K'. In this case, we can assume that p is not contained in the interior of an edge of the image any other patch. So, a neighborhood of p in K is covered by finitely many tiles of Δ_s and either 1, 2, or 4 patch images, each of which has p as a vertex. In any case, γ must exit this neighborhood, and some initial portion of γ makes a crosscut in one of the patch images. Now we pull back as before.

This exhausts the possibilities. ♠

The rest of the proof involves ruling out the existence of crosscuts associated to octagonal parameters.

7.6 The Proof Modulo one Case

Lemma 7.6 *Suppose that $s > 1/2$ is an octagonal parameter and K_s is a symmetric piece with a crosscut. Then the crosscut cannot contain an acute vertex of K_s.*

Proof: Suppose the crosscut contains a vertex p. Using the various kinds of bilateral symmetry discussed in §2.7 we reduce to the case when p is the bottom left vertex of X_s and K_s is one of B_s or P_s. (These are the two pieces having this point as a vertex.)

By Corollary 5.2, there are infinitely many octagons wedged into X_s. These octagons shrink down to p, and isolate p from the rest of S_s. So, p is its own connected component of S_s. ♠

We call a crosscut of K_s *acute* if it contains points p, q, where p lies in the interior of a short side of K_s and q lies in the interior of a long side of K_s. We use this name, because we think of a path joining p to q and subtending one of the acute angles.

Lemma 7.7 (Acute) *Suppose that $s > 1/2$ is an octagonal parameter and K_s is a symmetric piece with a crosscut. Then the crosscut cannot be acute.*

Proof: We prove this in the next section. ♠

Say that a *right crosscut* is a crosscut of K_s which contains the right-angled vertex of K_s.

Lemma 7.8 *Suppose that $s > 1/2$ is an octagonal parameter and K_s is a symmetric piece with a crosscut. The crosscut cannot contain the right-angled vertex of K_s.*

Proof: Let C be a crosscut which supposedly contains the right-angled vertex p of K_s. We will give the proof when C is a path, and then the Chain Trick handles the general case. Since s is octagonal, there are arbitrarily large choices of n for which $u = R^n(s) > 1/2$, If we choose n sufficiently large, then by the Covering Lemma we can find a patch (ψ_j, K_j, u) so that $K' = \psi_j(K_j)$ has p as a vertex, shares two edges with K, and has diameter (say) $\epsilon/10$. This is shown in Figure 7.2.

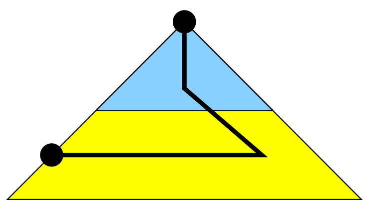

Figure 7.2: The pieces K and K'.

But then C must exit K' from the hypotenuse. The maximal initial portion of C contained in K' is an acute crosscut (with respect to either acute angle). But, Lemma 7.7 says that these cannot exist. ♠

Before we give the final argument, we single out some important points.

The Proper Nesting Property: Suppose $K' = \psi_j(K_j) \subset K_s$ is the image of some patch which arises in the conclusion of the Covering Lemma for the parameters s and u. If K' and K_s share a vertex, then this vertex has the same type (acute or right) with respect to both triangles. One sees this just by inspecting the equations used in the proof of Lemma 4.6. Call this the *proper nesting property*.

Images of Crosscuts: If (K_j, ψ_j, u) is a patch for the parameter s, and K_j has a right crosscut, then some connected subset $S_s \sqcap K'$ contains points on the interiors of both the short edges of $K' = \psi_j(K_j)$. We will abuse our terminology and say that K' has a crosscut, even though technically K' is a similar copy of a symmetric piece associated to the different parameter u.

With this terminology, a crosscut for K' is a subset of S_s.

We call a crosscut of K_s *right* if it contains points p, q, where p and q respectively lie in the interiors of the two short sides of K_s. Modulo proving Lemma 7.7, we have ruled out all types of crosscut except right crosscuts. So, if Statement 3 of the Main Theorem is false, then there is an infinite sequence of octagonal parameters $s_1, s_2, ...$ with the following properties.

- $s_{k+1} = R^{n_k}(s_k)$ for some $n_k > 0$.
- Some symmetric piece K_{s_k} has a right crosscut for $k = 1, 2, 3...$
- The crosscut associated to K_{s_k} is disjoint from a neighborhood of the right-angled vertex of K_{s_k}. (Otherwise, we reduce to the previous case.)

Once one symmetric piece K_s has a crosscut, all similar copies of K_s have crosscuts. Moreover, by Lemma 4.6, each similar copy of a symmetric pieces for the parameter s_j contains a similar copy of a symmetric piece for the parameter s_k as long as $k > j$.

Using the proper nesting property together with the Pidgeonhole Principle, we can fnd two nested (similar copies of) symmetric pieces which both have right crosscuts. We can arrange that the smaller piece is so small that its crosscut is completely disjoint from the crosscut of the larger piece, as shown in Figure 7.3.

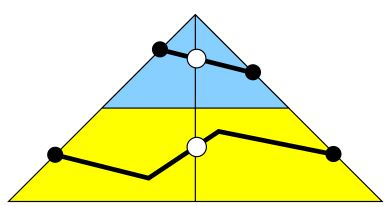

Figure 7.3: Nested symmetric pieces

Call s the parameter associated to the larger of the two symmetric pieces. By construction, both the crosscuts we have found must cross the line of symmetry of K_s. Moreover, both crosscuts are subsets of S_s. Hence, S_s crosses the line of symmetry for K_s at least twice. This contradicts the Pinching Lemma.

The only way out of the contradiction is that Statement 3 of the Main Theorem is true. This completes the proof of the Main Theorem, modulo the proof of Lemma 7.7.

7.7 No Acute Crosscuts

Here we complete the proof of the Main Theorem by establishing Lemma 7.7. If some symmetric piece K_s has an acute crosscut, then we can use bilateral symmetry to reduce to the case when S_s has a connected subset C containing a point on the bottom edge of X_s and a point on the left edge of X_s. We abbreviate this by saying that the parameter s has an acute crosscut. At this point we no longer care about the condition that $s > 1/2$. As above, we will treat the case when C is a path; the Chain Trick handles the general case.

Lemma 7.9 *If an octagonal parameter s has an acute crosscut, then there is another octagonal parameter u which has an acute crosscut not entirely contained in Z_s.*

Proof: If s has an acute crosscut contained in Z_s and $t = R(s)$ also has an acute crosscut. If the acute crosscut for t lies in Z_t we can repeat the procedure. Every one or two steps of the procedure, the distance between the endpoints of the crosscut increases by a factor of at least $\sqrt{2}$. So, eventually we reach a stage where the crosscut cannot lie in Z_s. ♠

Say that an *egregious crosscut* is a connected subset of S_s which contains a point in the top edge of X_s and a point on the bottom edge of X_s.

Lemma 7.10 *If $s < 1/2$ has an acute crosscut which does not lie in Z_s then $R(s)$ has an egregious crosscut.*

Proof: Define

$$Z_s^* = (Z_s^0 \cup R_D(Z_s^0)) \cap X_s \qquad (26)$$

Figure 7.4 shows this set.

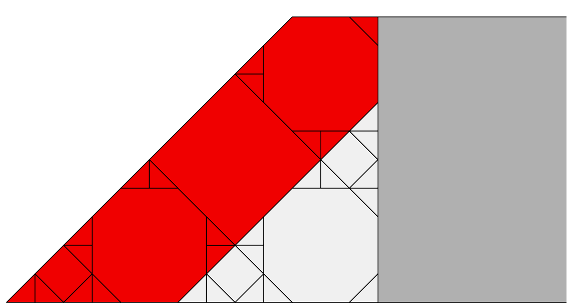

Figure 7.4: The set Z_s^* for $s = 5/13$.

C starts out on the left edge of Z_s^* and must eventially reach the bottom edge of X_s. But then C must cross the right edge of Z_s^*. By symmetry, the left branch ϕ_s^0 of the map ϕ_s from Theorem 2.5 extends to all of of Z_s^*. When we pull back the maximial initial portion of C lying in Z_s^*, we get an egregious crosscut for t. ♠

Lemma 7.11 *If $s > 1/2$ has an acute crosscut which does not lie in Z_s, then one of $R(s)$ or $R^2(s)$ has an egregious crosscut.*

Proof: Let K be the layering constant for s. When $K > 1$ we define

$$Z_s^* = (Z_s^0 \cup R_V(Z_s^0)) \cap X_s \tag{27}$$

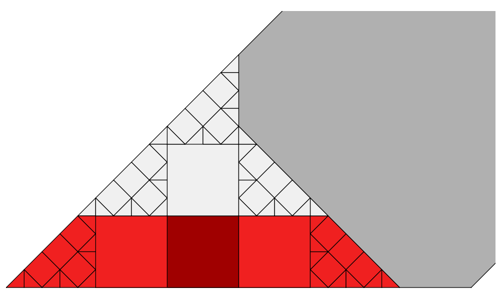

Figure 7.5: The set Z_s^* for $s = 11/13$.

If C starts out above Z_s^* then C must cross both the top and bottom edges of Z_s^* to reach the bottom edge of X_s. If C starts out in Z_s^* but does not lie entirely in Z_s^0, then C must exit the top of Z_s^*. The problem is that C cannot penetrate through the *dark red) central tile of Z_s^0. So, in this case as well, C crosses both the top and the bottom of Z_s^*. Moreover, C must cross both sides on the same side (left or right) of the central tile of Z_s^0. Reflecting in V, we can assume that C crosses both sides of Z_s^* to the left of the central tile of Z_s^0. But then some connected subset of C lies in Z_s^0 and contains points both on the top edge and the bottom edge. Now we pull back by ϕ_s, as above.

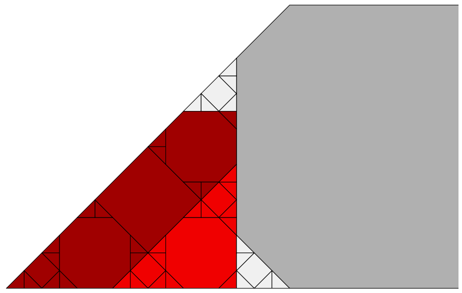

Figure 7.6: The set Z_s for $s = 11/13$.

Now suppose that $K = 1$. Figure 7.6 shows a representative example. The red set is Z_s^0. The dark red subset is a similar copy of Z_t^*, defined in Equation 26. Call this similar copy Ω. If C exits Z_s^0 then one of two things must happen.

1. C contains points on the left and right diagonal edges of Ω.

2. C contains points on the top edge and right diagonal edge of Ω.

In the first case, we proceed as in Lemma 7.10, and get an egregious crosscut for $u = R(t) = R^2(s)$. In the second case, we also proceed as in Lemma 7.10, but we get something different: After reflecting through the origin, we get a connected set $C' \subset S_u$ which has points on the bottom edge of X_u^0 and some point lying above the line H of symmetry. (The point is that C crosses the diagonal midline of Ω.) But then $C'' = C' \cup R_H(C')$ is an egregious crosscut for u. ♠

Combining what we have proved so far, we get the following.

Corollary 7.12 *It some octagonal parameter s has an acute crosscut, then some octagonal parameter $u < 1/2$ has an egregrious crosscut.*

Lemma 7.13 *If $s < 1/2$ has an egregious crosscut then $t = R(s) < 1/2$ and t has an egregious crosscut.*

Proof: When $s < 1/2$ and $R(s) > 1/2$, it follows from Theorem 2.5 (or from a direct calculation) that there is a large octagon which separates the top edge of X_s^0 from the bottom edge of X_s^0. See Figure 7.7.

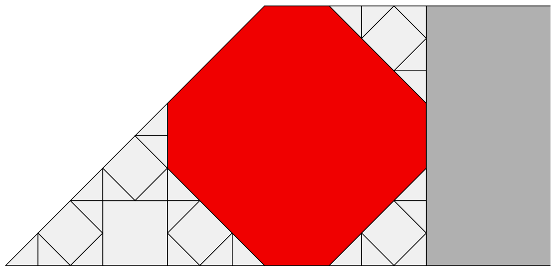

Figure 7.7: The large octagon.

Now we know that $R(s) < 1/2$. In this case, C has to cross the set

$$Z'_s = R_D(Z^0_s) \cap X_s. \tag{28}$$

C starts out in the top edge of Z'_s and exits through the bottom edge.

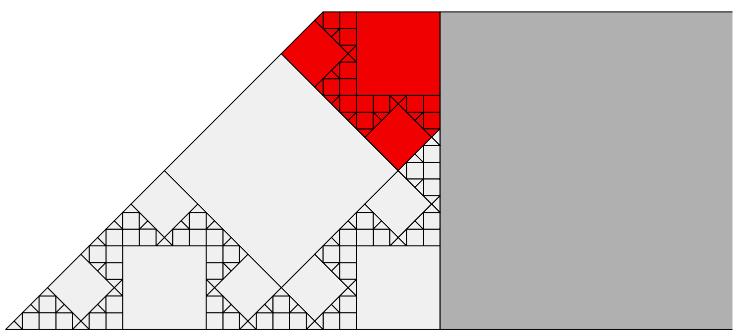

Figure 7.8: Z'_s (red) for $s = 19/52$.

The set

$$C' = \phi_s^{-1} \circ R_D(C) \tag{29}$$

connects a point on the bottom edge of X_t with a point that lies above the line H of symmetry. In this case (as in the proof of Lemma 7.11) $C' \cup R_H(C')$ is an egregious crosscut for t. ♠

Iterating Lemma 7.13, we see that $R^n(s) < 1/2$ for all n when s has an egregious crosscut. So, s cannot be octagonal in this case! If we assume that Statement 3 of the Main Theorem is false, then we have reached a contradiction.

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