# Links inside Straight-Edge Embeddings of Complete Bipartite Graphs

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**Abstract**

We prove the following theorem. Let $L$ be any link. There is some $N = N_L$ such that every straight-edge embedding of the complete bipartite graph $K_{N,N}$ contains a finite union of cycles having link type $L$. This result builds on the ideas of S. Negami, who proved the analogous result for complete graphs. Most of our motivation for this paper is to give a simpler proof of Negami's Theorem.

## 1 Introduction

In his 1991 paper, [**N**], Seiya Negami proved a beautiful theorem [1] about linearly embedded complete graphs.

**Theorem 1.1 (Negami)** *Let $L$ be any link. Then there is some integer $N = N_L$ with the following property. Any straight-edge embedding of the complete graph $K_N$ contains a finite union of cycles having the same link type as $L$.*

A *straight-edge* embedding of $K_N$ is an embedding into $\boldsymbol{R}^3$ such that all the edges are realized by straight line segments. All that is required is that the vertices of $K_N$ be in general position – i.e., no 4 coplanar. To say

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[1]It is worth mentioning that Negami also proves a version of his result for knotted graphs, but for ease of exposition we ignore this case.

that a union of cycles has *link type* $L$ is to say that some continuous motion deforms the union of polygons into $L$ without causing any of the strands to cross themselves. In short, the union of polygons is *isotopic* to $L$. Negami's paper has a history of the problem of finding knots and links in embedded complete graphs.

The purpose of this paper is to prove the following result.

**Theorem 1.2** *Let $L$ be any link. Then there is some integer $N = N_L$ with the following property. Any straight edge embedding of the complete bipartite graph $K_{N,N}$ contains a finite union of cycles having link type $L$.*

Theorem 1.2 grew out of our attempts to understand Negami's original paper, and the proof we give of Negami's Theorem (while mainly following his ideas) is considerably simpler. Theorem 1.2 is a strengthening of Negami's Theorem because the complete bipartite graphs are subgraphs of the complete graphs, whereas even the complete graph $K_3$ is never the subgraph of a complete bipartite graph. Strictly speaking, not all the steps we use to prove Negami's Theorem are needed for the proof of Theorem 1.2, but they do put our proof of Theorem 1.2 in context.

I would like to thank Ramin Naimi for telling me about Negami's Theorem, and also for helpful and interesting discussions about spatially embedded graphs.

# 2 Proofs of the Results

## 2.1 Ramsey's Theorem

For ease of exposition, we will state Ramsey's Theorem just for 2-colorings. Let $S_k(A)$ denote the set of all $k$-element subsets of a finite set $A$.

**Theorem 2.1 (Ramsey)** *Let $k$ and $n$ be given positive integers. There exists some $N = N(k,n)$ with the following property. Suppose that every element of $S_k(A)$ has been colored either red or blue. Then $A$ has an $n$-element subset $A'$ such that every element of $S_k(A')$ has the same color.*

The numbers $N(2,n)$ are the classic Ramsey numbers. See [**W**, §8.3] for a discussion and proof of Ramsey's Theorem.

## 2.2 Improving the Position

Negami's Theorem is a close cousin of the following result.

**Lemma 2.2** *Given an integer $n$ there is some other integer $N = N_n$ such that any $N$ points in the plane in general position contain the vertex set of a convex $n$-gon.*

**Proof:** Assume $n \geq 5$ and let $N = N(4, n)$. Let $A$ be the $N$-element set. We color an element of $S_4(A)$ red if the corresponding points form a convex quadrilateral, and otherwise blue. Note that the copy of $K_4$ using the vertices of a blue quadruple is embedded. By Ramsey's Theorem, there is an $n$-element subset $A' \subset A$ such that $S_4(A')$ has a monochrome coloring. If all these points are blue, then one can give a planar embedding of $K_n$ using the vertices of $A'$. But $K_5$ is not planar. Hence all elements of $S_4(A')$ are red. Hence $A'$ is convex. ♠

Let $\pi : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ be projection to the plane. Say that a subset $S \subset \boldsymbol{R}^3$ is *clean* if $\pi(S)$ is contained in the graph of a convex function. This means that the points of $\pi(S)$ are the vertices of a convex polygon, and also may be ordered from left to right consecutively.

**Corollary 2.3** *Given any integer $n$, there is some integer $N$ with the following property. If $S$ is a subset of $N$ general position points in $\boldsymbol{R}^3$, then (after rotating $S$ if necessary) some $n$-element subset $S' \subset S$ is clean.*

**Proof:** Let $N = N(4, 2n)$. By the previous result, there is a $2n$-element subset $S'$ such that $\pi(S)$ is the vertex set of a convex $2n$-gon. But then we can divide this $2n$-gon half and rotate so that $n$ of the points lie on the graph of a convex function. ♠

We call a straight-edge embedded complete graph *clean* if its vertices form a clean set. Let $\Gamma$ be a clean complete graph. We orient the edges of $\Gamma$ so that they point from left to right, when projected into the plane. In other words, the tail vertex of each edge has smaller $x$-coordinate than the head vertex. We call a pair of edges *crossing* if their planar projections cross, and *positive* if the crossing is positive in the sense of Figure 1 below. We call $\Gamma$ *positive* if every pair of crossing edges is positive.

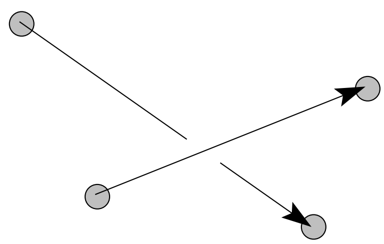

**Figure 1:** A positive crossing

**Lemma 2.4** *Given any $n$ there is some $N$ with the following property: If $\Gamma$ is a clean complete graph of size $N$, then (up to mirror reflections) $\Gamma$ contains a positive clean complete graph $\Gamma'$ of size $n$.*

**Proof:** Let $N = N(4, n)$ and let $A$ be be a clean set of $N$ points. Given an element $\beta \in S_4(A)$, there is a unique way to pair the corresponding points so that the edges cross when projected into the plane. We color $\beta$ red if these edges make a positive crossing, and otherwise blue. Ramsey's Theorem gives us a set $A' \subset A$ of size $n$ such that of $S_4(A')$ has a monochrome coloring. If the color is red, we are done. If the color is blue, we reflect the picture in the $xy$ plane. ♠

## 2.3 The Twisted Cubic

Below we will prove the following claim. Given any link $L$, there is some $N = N_L$ such that any positive clean complete graph of size $N$ contains a cycle with link type $L$. Call this claim $C(+)$. In view of Lemma 2.4, and the fact that $C(+)$ is supposed to hold for both a link and its mirror image, Negami's Theorem follows from $C(+)$.

Let $K_n$ denote the complete graph on $n$ vertices. Any two clean positive embeddings of $K_n$ are equivalent in the following sense: They contain precisely the same links: The obvious bijection between two such embeddings is such that the corresponding cycles in each one have the same planar diagrams. Given this equivalence, it is useful to have a nice model for a clean positive complete graph. The twisted cubic provides such a model. The twisted cubic is the curve $X$ with parametric equations $(t, t^2, t^3)$. This curve projects to the parabola $(t, t^2)$. Any collection of $n$ points on $X$ gives rise to a clean positive embedding of $K_n$. So, for the purposes of proving $C(+)$, we just have to show that any link can be realized as a polygon having vertices on the twisted cubic. This formulation is made in Negami's paper.

### 2.4 Realizing the Link

Let $f(x,y,z)=x$ denote the map which takes the first coordinate. Say that a smooth link $L$ is in *bridge position* if if $f(L)=[0,1]$ and the only critical points of $f$ are either global minima, namely $f^{-1}(0)$, and global maxima, namely $f^{-1}(1)$. When $L$ is in bridge position, $L$ is realized as a bipartite graph where the (not necessarily straight) edges connect minima to maxima. It is a well-known result that every link can be put in bridge position. Intuitively, you just clasp all the minima of the link with your left hand, and all the maxima with your right hand – then you pull the link tight, like a rubber band. So, as a first step to proving $C(+)$, we put the given link in bridge position.

Let $L$ be a link in bridge position. Thinking of $L$ as a bipartite graph, we orient each of the strands of $L$ from left to right. We say that $L$ is in *positive bridge position* if all the crossings are positive. Following Negami, we prove:

**Lemma 2.5** *Every link can be placed in positive bridge position.*

**Proof:** Scanning the link from left to right, you look for the first negative crossing. Assuming you have found a negative crossing, you give the right half of the link a twist while keeping (most of) the left half fixed. This twist has the effect of removing the negative crossing at the expense of adding some new positive crossings. Just do this finitely many times to eliminate all negative crossings. ♠

At this point, our proof diverges from what Negami does. Given a link in positive bridge position, we can spread out the crossings so that they appear sequentially as on the left side of Figure 2. The portion between each set of vertical lines is one of a small number of standard types. We will call such a portion a *unit*. (The vertical lines are not part of the link.) The crossings are all understood to be positive. The link [2] below has 4 units.

Each individual unit of the link $L$ can certainly be realized as a collection of arcs with endpoints on the twisted cubic. Moreover, we can realize each unit individually in such a way that it lies in a thin tubular neighborhood of a single segment. Then we can concatenate the individual units, in a zig-zag pattern, as shown on the right hand side of Figure 2. The positivity guarantees that the segments of a given unit all cross over, or all cross under, the

[2]The example in Figure 2 is the unknot, but it serves to illustrate our general method.

segments of an adjacent unit. For this reason, there is a fairly obvious isotopy which just "unfolds" the realization back into the link $L$. This completes the proof of Negami's Theorem.

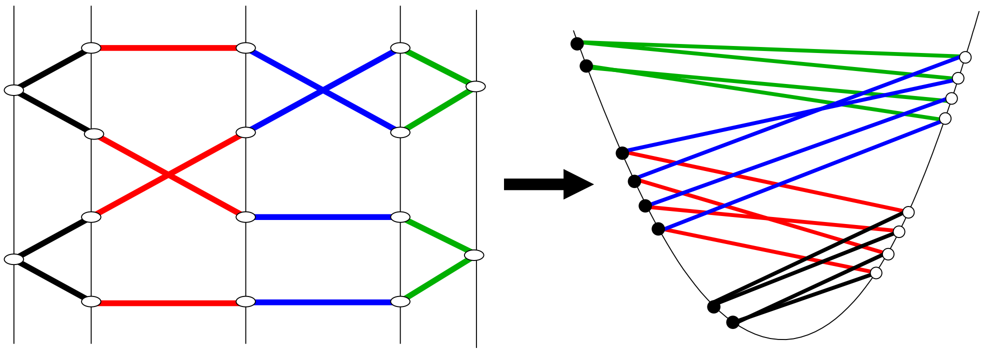

**Figure 2:** Realizing a link on the twisted cubic

## 2.5 A Corollary of Ramsey's Theorem

The following variant of Ramsey's Theorem concerns the coloring of certain elements of $S_4(A \cup B)$, namely those which have 2 elements in common with each of $A$ and $B$.

**Lemma 2.6** *For any integer $n > 0$ there is some $N = N_n$ with the following property. Suppose that $A$ and $B$ are two disjoint $N$-element sets. Suppose every element of $S_2(A) \times S_2(B)$ bas been colored red or blue. Then there exist $n$-element subsets $A' \subset A$ and $B' \subset B$ such that every element of $S_2(A') \times S_2(B')$ has the same color.*

**Proof:** Let $e_1, ..., e_M$ be the list of unordered pairs of vertices in $A$. Let $B_0 = B$. Assuming that $B_k$ is defined, let $B_{k+1}$ denote the largest subset of $B_k$ such that every element of $S_2(e_{k+1}) \times S_2(B_{k+1})$ has the same color $c_k$. If we choose $N$ large enough, then repeated applications of Ramsey's Theorem guarantee that we can make $B_M$ have cardinality at least $n$. At the same time, we can choose $N$ large enough so that there is some $n$-element set $A' \subset A$ such that $c_k$ is the same for all $e_k \subset A'$. Let $m$ be the largest index such that $e_m \subset A'$. Let $B'$ denote any $n$ element subset of $B_m$. Then $A'$ and $B'$ satisfy the conclusion of the lemma. ♠

### 2.6 Proof of Theorem 1.2

Suppose that $X = A \cup B$ is a union of $2N$ points in $\boldsymbol{R}^3$, with $N$ points each in $A$ and in $B$. Let $K_{N,N}$ be the straight-edge bipartite graph defined by $X$. Let $\pi$ denote projection to $\boldsymbol{R}^2$. If necessary, we perturb, take subsets, swap $A$ and $B$, and translate so that the points of $\pi(X)$ are in general position and $\pi(A)$ lies to the left of the $y$-axis and $\pi(B)$ lies to the right of the $y$-axis. We orient the edges of $K_{N,N}$ from left to right, so that their tails are in $A$ and their heads are in $B$.

**Lemma 2.7** *For any $n$ there is some $N = N_n$ with the following property. Suppose $X = A \cup B$ is the general position union of two disjoint $N$-element subsets of points in $\boldsymbol{R}^3$. Then there are $n$-element subsets $A' \subset A$ and $B' \subset B$ such that every quadrilateral formed from two points of $\pi(A')$ and two points of $\pi(B')$ is convex.*

**Proof:** Assume $n \geq 3$. We color an element of $S_2(A) \times S_2(B)$ red if the corresponding 4 points project to a convex quadrilateral, and otherwise blue. Lemma 2.6 gives us $n$-element sets $A' \subset A$ and $B' \subset B$ such that every element of $S_2(A') \times S_2(B')$ has the same color. If all these elements are blue, then the points $\pi(A')$ and $\pi(B')$ can be used as the vertices of a straight-edge planar embedding of $K_{n,n}$. But $K_{3,3}$ is not planar. Hence, all elements of $S_2(A') \times S_2(B')$ are red. ♠

Let $A = \{a_1, ..., a_N\}$ and $B = \{b_1, ..., b_N\}$. Consider, for $i < j$ and $k < l$, the 4-tuple $(a_i, a_j, b_k, b_l)$. By Lemma 2.7 we can reduce to the case where these points necessarily project to a convex quadrilateral. This means that either $\pi(a_i b_k)$ and $\pi(a_j b_l)$ cross or $\pi(a_i b_l)$ and $\pi(a_j b_k)$ cross, but not both.

Another application of Lemma 2.6, followed possibly by reversing the ordering of $A$, reduces to the case where $\pi(a_i b_l)$ always crosses $\pi(a_j b_k)$. Yet another application reduces to the case where such crossings are all positive in the sense of Figure 1. A final application, followed possibly by the reversal of both the ordering on $A$ and the ordering on $B$, reduces to the case where $\pi(a_i b_k) \cap Y$ always lies below $\pi(a_j b_l) \cap Y$. Here $Y$ denotes the $y$-axis.

We now construct out realization of the link $L$ in a manner similar to what is shown on the right hand side of Figure 2. Assume that $L$ is given as a link in positive bridge position with $k$ local minima and $k$ local maxima. We realize the first unit of $L$ using vertices $a_1, ..., a_k$ and $b_1, ..., b_{2k}$. We realize the second unit using vertices $b_1, ..., b_{2k}$ and $a_{k+1}, ..., a_{3k}$. We realize the third unit using vertices $a_{k+1}, ..., a_{3k}$ and $b_{2k+1}, ..., b_{4k}$. And so on. Thanks to the crossing properties of $K_{N,N}$, our realization $L'$ has the following general features.

- $L'$ decomposes into units, each having a projection isomorphic to the corresponding unit of $L$.
- Each unit of $L'$ either crosses entirely over or entirely under the adjacent unit, and non-adjacent units have disjoint projections.
- The successive units of $L'$ form a zig-zag pattern, as in Figure 2, and generally move upwards in terms of how they cross the $y$-axis.

From these features, we see that $L'$ is isotopic to $L$. This completes the proof of Theorem 1.2.

# 3 References

[**N**] S. Negami, *Ramsey Theorems for Knots, Links, and Spatial Graphs*, Transactions of the American Math Society (1991) pp 527-541

[**W**] D. B. West, *Introduction to Graph Theory, 2nd Ed.*, Prentice Hall (2001).