Erratum: (By Rich Schwartz) Lemma 2.10 of our paper

[**OST** V. Ovsienko, R. Schwartz, S. Tabachnikov, *The Pentagram Map: A Discrete Integrable System*, Communications in Mathematical Physics 2010

claims that $\{O_i, O_j\} = \{O_i, E_j\} = \{E_i, E_j\} = 0$ for all relevant indices i and j. Here $\{\ ,\ \}$ is the Poisson bracket in [**OST**].

Description of the Bracket: Given monomials A and B, we form a bipartite graph, where the top vertices and the bottom vertices are both indexed by the set $\{1, ..., 2n\}$. We join the top vertices a_i to the bottom vertices $b_{i\pm2}$ iff x_i appears in A and $x_{i\pm2}$ appears in B. Indices are reckoned cyclically, as usual. We label the edge joining a_i to $b_{i\pm2}$ with $(\pm)$ if i is even and with $(\mp)$ if i is odd. Then $\{A, B\}/AB$ is the number of $(+)$ signs minus the number of $(-)$ signs.

We prove first that $\{O_i, O_j\} = 0$. The only monomials which can appear in $\{O_i, O_j\}$ have exponents in the set $\{1, 2\}$. In our proof, we sometimes view the monomial μ as a *mapping* $\mu : \{1, ..., 2n\} \to \{0, 1, 2\}$. Here $\mu(i)$ is the exponent of x_i in μ. The *support* of μ (as a map) is exactly the set of indices of variables which appear in μ (as a monomial). We define $\{O_i, O_j; \mu\}$ to be the coefficient of μ in $\{O_i, O_j\}$. We call μ *good* if $\{O_i, O_j, \mu\} = 0$ for all indices i, j. We will prove that all monomials are good.

We say that μ *decomposes* into μ_1 and μ_2 if (as monomials) $\mu = \mu_1\mu_2$, and (as maps) the supports of μ_1 and μ_2 are separated by at least 2 empty spaces, in the cyclic sense. If we cannot factor μ this way, we call μ *indecomposible*. Below we prove the following results.

Lemma 0.1 *If μ decomposes into μ_1 and μ_2, and both μ_1 and μ_2 are good, then μ is good.*

Lemma 0.2 *Suppose μ is indecomposible and (A, B) contributes nontrivially to $\{O_i, O_j, \mu\}$. Then A and B have the same weight.*

Let μ be a monomial. By Lemma 0.1, it suffices to assume μ is indecomposible. If some (A, B) contributes nontrivially to $\{O_i, O_j, \mu\}$ then A and B have the same weight. Hence (B, A) also contributes to $\{O_i, O_j, \mu\}$. But $\{A, B\} = -\{B, A\}$ and the two contributions cancel. Hence μ is good.

Proof of Lemma 0.1: For any monomial F, we let F_1 (respectively F_2) denote the monomial obtained from F by setting to 1 all the variables having indices in the support of μ_2 (respectively μ_1.) Consider the example when $\mu = x_1x_5x_7$, which decomposes into $\mu_1 = x_1$ and $\mu_2 = x_5x_7$. If $F = x_1x_5$ then $F_1 = x_1$ and $F_2 = x_5$.

Let $S(i, j, \mu)$ denote the set of pairs (A, B) contributing to the sum $O(i, j, \mu)$. Here A has weight i and B has weight j and $AB = \mu$. Let $S(i, j, \mu, , i', j') \subset S(i, j, \mu)$ denote the set of pairs (A, B) such that A_1 has weight i' and B_1 has weight j'. Continuing with our example, $S(1, 2, x_1x_5x_7, 0, 1)$ contains the pairs (x_5, x_1x_7) and (x_7, x_1x_5).

By construction

$$O(i, j, \mu) = \sum_{i' \leq i,\ j' \leq j} O(i, j, \mu, i', j'), \tag{1}$$

where

$$O(i, j, \mu, i', j') = \sum_{(A,B) \in S(i,j,\mu,i',j')} \frac{\{A, B\}}{AB}. \tag{2}$$

There is a bijection

$$S(i', j', \mu_1) \times S(i - i', j - j', \mu_2) \to S(i, j, \mu, i', j') \tag{3}$$

given by that map $((A_1, B_1), (A_2, B_2)) \to (A_1A_2, B_1B_2)$. From the large separation between the supports of A and B, we have $\{A_i, B_{3-i}\} = 0$. Hence, by Leibniz's rule,

$$\{A_1A_2, B_1B_2\} = \{A_1, B_1\}A_2B_2 + \{A_2, B_2\}A_1B_1. \tag{4}$$

Letting $|S|$ denote the cardinarlity of a set S, we see from Equation 4 that

$$O(i, j; \mu; i', j') = |S(i-i'; j-j')|O(i', j'; \mu_1) + |S(i', j'; \mu_1)|O(i-i', j-j'; \mu_2) = 0. \tag{5}$$

Summing over all i', j' gives $O(i, j; \mu) = 0$. ♠

Proof of Lemma 0.2: This is trivial if the support of μ is at most 3 indices, so we suppose otherwise. Say that μ *has* $a_1...a_k$ if there are k consecutive indices $i_1, ..., i_k \in \{1, ..., 2n\}$ such that $\mu(i_j) = a_j$ for $j = 1, ..., k$. Call i_j the *place* of a_j. We say that a *unit* of μ is a maximal string of nonzero digits which μ has, in the sense just defined. Observe the following.

1. μ cannot have 2 in an even place. Suppose $\mu(4) = 2$. Then $x_3x_4x_5$ appears in both A and B, and so neither A nor B contains x_k for $k = 0, 1, 2, 6, 7, 8$. Since the support of μ is not just $\{3, 4, 5\}$, we get μ decomposible, a contradiction. Similarly, μ cannot have 020.

2. μ cannot 10 or 01 if the place of the 1 is even. Likewise, if μ has $111c$ or $c111$ then $c = 0$. In both cases, the problem is that one of A or B would have an even-indexed variable but not one of the adjacent odd-indexed variables.

3. If μ has $2cd$ or $dc2$ then $c = d = 0$ or $d = c = 1$. The previous observations rule out $c = 2$ and 210 and 012. The case $d = 2$ forces $c = 2$, and $d = 1$ forces $c = 1$.

These observations imply that the only possible units are 1, 111, 211, 112, and 11211, and that adjacent units are separated by a single 0, and that 211 (respectively 112) cannot have an adjacent unit on its left (respectively right).

If μ assigns 0 to two consecutive indices, then there is a canonical way to define the leftmost unit; otherwise we choose arbitrarily. Scanning the units from left to right, we create a word $w(A, B)$, using letters a and b, as follows. For each unit 211 we write ab (respectively ba) if the variables corresponding to 111 belong to B (respectively A). We do the mirror image for 112. For each unit 1 or 111 we write a (respectively b) if the corresponding variables appear in A (respectively B). For each unit 11211 we write ab (respectively ba) if the first 3 variables belong to A (respectively B). We can recover A and B from μ and $w(A, B)$. Here is the key point. Since A and B are both admissible, the letters in $w(A, B)$ alternate.

Suppose (A, B) is a minimal counterexample, in terms of weight. Suppose μ has the unit 11211. Let μ' denote the indecomposible monomial having the same units as μ, in the same order, but with a single 11211 omitted. We omit and collapse, so to speak. We define (A', B'), uniquely, so that $w(A', B')$ is obtained from $w(A, B)$ by omitting either ab or ba. It follows from our description of the bracket that $\{A, B\}/AB = \{A', B'\}/A'B'$. See the picture. By construction A' and B' have the same weight as each other. In short, (A', B') is a smaller counterexample. Similar arguments show that μ cannot contain 112 or 211 or consecutive units from the set $\{1, 111\}$. Hence μ has 1 unit. But there are no 1-unit counterexamples. ♠

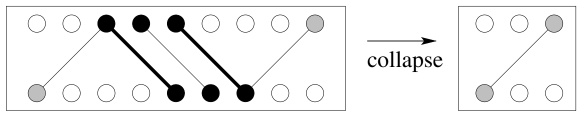

Figure 1: Collapsing the unit 11211.

It follows from the odd case and symmetry that $\{E_i, E_j\} = 0$. To prove $\{O_i, E_j\} = 0$, we use the same set-up as above. Lemma 0.1 works again, and gets us to the indecomposible case.

Lemma 0.3 *If μ is indecomposible and has 101 or 2 then no terms contribute to $\{O_i, E_j, \mu\}$.*

Proof: Let (A, B) be a supposedly contributing pair. Suppose μ has 101. If the 1s are in odd places, then B, an even admissible monomial, has the variable x_o but not x_{o-1} for some odd index o. This is a contradiction. A similar contradiction obtains if the places of the 1s are even.

Suppose the place of 2 is odd, say $\mu(5) = 2$. Then B contains $x_4x_5x_6$ and A contains x_5, but not both x_4 and x_6. Suppose neither x_4 nor x_6 appears in A. Then x_3 and x_7 appear in neither A nor B. Hence, $\mu(3) = \mu(7) = 0$. Since μ indecomposible and the support is not contained in just $\{5\}$, we must have $\mu(2) \neq 0$ or $\mu(8) \neq 0$. But, neither x_2 nor x_8 can belong to A or B. This is a contradiction. Suppose x_4 appears in A. Then x_3 appears in A and x_6 does not. Since x_2 does not appear in B, we have $\mu(2) = 0$. As in the previous case, $\mu(7) = 0$. Since μ is indecomposible and its support is more than just $\{4, 5\}$, either $\mu(1) \neq 0$ or $\mu(8) \neq 0$. Now we have the same contradiction as previously. The proof is the same when A contains x_6.

The same argument, with the roles of A and B reversed, works when the place of 2 is even. ♠

Now we know that μ has a single unit, consisting of a string of 1s. When we label each index in the support by an a or a b, indicating the monomial which contains the corresponding variables, the pattern must be one of $*aaabbbaaabbb...*$ or $*bbbaaabbbaaa...*$, with $*$ being either empty or a single a or b – the opposite of its neighbor. An inductive argument as above shows that $\{A, B\} = 0$ unless μ has an odd number of 1s and the pattern is not a palindrome. In the odd, non-palindromic case, the reversed pattern corresponds to a second, and different, term which cancels the first. For instance the terms $\{x_1x_5x_6x_7, x_2x_3x_4\}$ and $\{x_1x_2x_3x_7, x_4x_5x_6)\}$, corresponding to *abbbaaa* and *aaabbba*, cancel each other. This completes the proof.