Mostly Surfaces: Errata

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September 27, 2013

Current Date: 26 Sept 2013

This is a running list of errata for my book. In each section, I'll list the mistake and (when possible) offer some replacement text which fixes the mistake.

1 Section 8.5

1.1 The Problem

This section proves the Bolyai-Gerwein Theorem, but never states the result. So, I should add the statement of the theorem.

1.2 Replacement Text

We continue with the theme of polygon dissections. Here we prove the classic result that any two polygons having the same area are dissection equivalent. The formal definition is given below, but informally this means that one can cut the first polygon into finitely many pieces, using straight-edge cuts, and then reassemble the pieces to make the second polygon. This result is called the *Bolyai-Gerwein Theorem*...

(the rest of the text is the same.)

2 Section 17.4

2.1 The Mistake

The definition of a translation surface given here is incomplete, and Theorem 4.1 is not correct. What is technically known as a translation surface requires a slightly stronger definition than I have given. In particular, Theorem 4.1 is not correct. The easiest fix is to simply add the condition of Theorem 4.1 into the definition, and then omit Theorem 4.1 entirely.

2.2 Replacement Section 17.4

A *translation surface* is a Euclidean cone surface which admits a parallel vector field which is defined everywhere except at the cone points. By Exercise 3 above, the cone angles of a translation surface are all integer multiples of 2π.

At first it might seem that a Euclidean surface whose cone angles are all integer multiples of 2π must admit a parallel vector field, but this is not so. As Rick Kenyon pointed out to me, M. Troyanov constructed some counter examples. See "Les surfaces euclidienne a singularites coniques", by M. Troyanov, published in Enseign. Math (2) 32 (1986), 79-94. You might like to try to find some examples yourself without looking up Troyanov's article.

(After Theorem 4.1 is omitted, the rest of the text remains as is.)

3 Section 18.6

In the second paragraph, the sentence *The corresponding linear fractional transformation acting on* $\boldsymbol{H}^2$ *is a hyperbolic reflection* is inaccurate. The sentence should read *the corresponding hyperbolic isometry acting on* $\boldsymbol{H}^2$ *is a hyperbolic reflection.*

4 Section 24.4

4.1 The Mistake

The proof of Lemma 24.3 is not correct. Even though Lemma 24.4 is correct as stated, it does not always help to prove Lemma 24.3.

4.2 Replacment Section 24.4

Let P be a polygon whose edges are labelled $(+)$ and $(-)$. We say that P has a *good labelling* if the list of labels around its edges is good; that is, there at least 2 sign changes from $(+)$ to $(-)$. A quadrilateral with a good labelling must be labelled $(+,-,+,-)$, up to cyclic ordering.

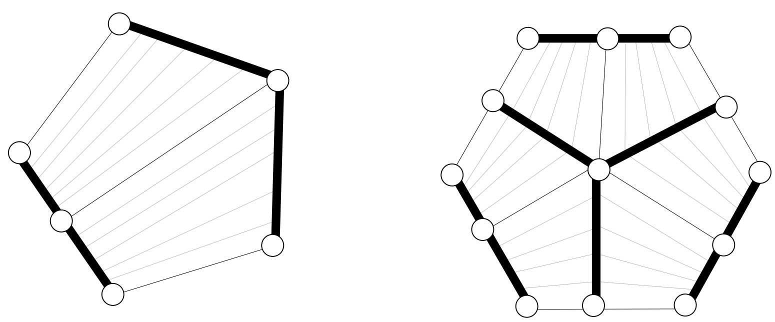

Figure 24.1. Adding edges and vertices and filling in with curves.

Figure 24.1 shows examples of how one can divide a polygon with a good labelling into quadrilaterals with good labellings. In the subdivision process, you are allowed to add both edges and vertices.

Exercise 6. Prove Lemma 4.1 below.

Lemma 4.1 *Suppose that P is a polygon with a good labelling, and all the edge labels are nonzero. Let P_v denote the vertex set of P. We can partition P into alternately labelled quadrilaterals, extending the labelling on P, such that the following is true:*

- *Every vertex has at least 3 edges incident to it.*
- *No vertex is incident to 3 consecutive $(+)$ edges.*

We think of P as a flat cone surface. Let P' denote P minus the vertices. We fill each quadrilateral of the partition with arcs which run perpendicular to the $(+)$ edges and parallel to the $(-)$ edges. These curves piece together across the $(+)$ edges to fill all of P'. The lines tangent to these curves define a continuous *line field* on P'.

To show that such a line field cannot exist, we imitate the proof of the Hairy Ball Theorem given in Section 9.6. To an oriented loop γ in P', we assign an integer $w(\gamma)$ which records how many times the line field turns clockwise relative to the tangent lines of γ as we travel once around γ. To fix the orientations, we identify P with $\boldsymbol{C} \cup \infty$ in such a way that 0 and ∞ are not vertices.

Let γ_0 be a small counterclockwise loop around the origin. Let γ_1 be a large counterclockwise loop which surrounds all the vertices of P. Let $\{\gamma_t | \ t \in [0,1]\}$ be a homotopy between γ_0 and γ_1. We imagine γ_t expanding from γ_0 to γ_1, as on the left hand side of Figure 9.3. As in the Hairy Ball Theorem, we can use the homotopy on the right hand side of Figure 9.3 to show that $w(\gamma_1) = -2$.

The value $w(\gamma_t)$ can only change when γ_t passes through one of the vertices of P. When γ_t passes through a vertex, we have

$$w(\gamma_{t+\epsilon}) = w(\gamma_{t-\epsilon}) + w(\beta) - 2, \tag{1}$$

where β is a small counterclockwise loop around the vertex. To see this, we arrange that $\gamma_{t-\epsilon}$ and $\gamma_{t+\epsilon}$ differ only in a small neighborhood of the vertex, so that $\gamma_{t+\epsilon}$ is obtained from $\gamma_{t-\epsilon}$ by "annexing" β. Note that the formula is obviously correct if the line field extends continuously across the vertex, because then $w(\beta) = 2$ and $w(\gamma_{t-\epsilon}) = w(\gamma_{t+\epsilon})$. This suggests that the formula correctly records the local analysis in general.

The conditions in Lemma 24.4 force $w(\beta) \geq 2$. The key point is that our curves never connect points on two consecutive $(+)$ edges emanating from the vertex. Since $w(\beta) \geq 2$, we have $w(\gamma_{t+\epsilon}) \geq w(\gamma_{t-\epsilon})$. But then $w(\gamma_1) \geq 2$, which is a contradiction.

(the rest of the text is the same)