On The Affine Shape of a Figure-Eight under the Curve Shortening Flow

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Abstract

We consider the curve shortening flow applied to a class of figure-eight curves, those with dihedral symmetry, convex lobes, and a monotonicity assumption on the curvature and its derivative. We estimate the shapes of the possible limits when (non-conformal) linear transformations are applied to the solution so as to keep the bounding box the unit square. Along the way we prove that suitably chosen arcs of our evolving curves, when suitably rescaled, converge to the Grim Reaper Soliton under the flow. Our Grim Reaper Theorem is an analogue of a theorem of S. Angenent, which Angenent proved in the locally convex case.

1 Introduction

We say that a smooth family $C : S^1 \times [0,T) \to \mathbf{R}^2$ of closed immersed plane curves is evolving according to *curve shortening flow* (CSF) if and only if for any point $(u,t) \in S^1 \times [0,T)$ we have

$$\frac{\partial C}{\partial t} = kN$$

where k is the curvature and N is the unit normal vector of the immersed curve $u \to C(u,t)$. We often abbreviate this curve as $C(t)$. In all cases, there is some time $T > 0$, called the *vanishing time*, such that $C(t)$ is defined for all $t \in (0,T)$ but not at time T.

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Some powerful results are known about this PDE. In [9], M. Gage and R. Hamilton prove that when $C(0)$ is convex the curve $C(t)$ (which remains convex) shrinks to a point as $t \to T$ and moreover there is a similarity S_t such that $S_t(C(t))$ converges to the unit circle. See also [10] and [11]. In [12], M. Grayson proves that if $C(0)$ is embedded then there is some time $t \in (0,T)$ such that $C(t)$ is convex. Thus, the combination of these two results says informally that the curve-shortening flow shrinks embedded curves to round points.

In [3], S. Angenent proves that if $C(0)$ is immersed and with finitely many self-intersections, then the number of self-intersections is monotone non-increasing with time. In the case of a Figure-8, a smooth loop with one self-intersection, M. Grayson proves two things:

- If one of the two lobes of the figure-8 has smaller area than the other, then this lobe shrinks to a point before the vanishing time. Then the flow can be continued through the singularity and it turns into the embedded case.

- If the lobes have equal area, the double point does not disappear before the vanishing time T, and the isoperimetric ratio of $C(t)$ tends to ∞ as $t \to T$.

Grayson conjectures [13] that in the second case, the figure-8 converges to a point under the curve-shortening flow, but this is as yet unresolved. In case $C(0)$ has 2-fold rotational symmetry, it follows from Corollary 2 of [6] that $C(t)$ does shrink to a point (the double point) as $t \to T$. In a related direction, the papers [1], [7], and [14] discuss self-similar solutions to the CSF. These shrink to a point and retain their shape.

We work with figure-8 curves that have convex lobes and 4-fold dihedral symmetry. We normalize so that the coordinate axes are the symmetry axes and that the x-axis intersects the curve in 3 points. Thus, our curves look like ∞ symbols. In [3] it is proved that if $C(0)$ has convex lobes then so does $C(t)$ for all $t \in (0,T)$.

Let $C_+(t)$ denote the righthand lobe of $C(t)$. We define $\kappa(\theta,t) > 0$ to be the curvature of $C_+(t)$ at the point where the tangent line makes an angle θ with the x-axis. We measure this angle in such a way that the top half of $C_+(t)$ is parametrized by $\theta \in (-\alpha(t), \pi/2]$, where $\alpha(t)$ is the tangent angle at the origin. Let $\kappa_\theta = \partial\kappa/\partial\theta$, etc. Computing the time evolution of κ, we have

$$\kappa_t = \kappa^2(\kappa + \kappa_{\theta\theta}). \tag{1}$$

See [10] and [11] for a proof. See also [2] and [4]. We note that the curve satisfying $\kappa(\theta) = \sin(\theta)$, for $\theta \in (0,\pi)$ is a stationary solution for Equation 1. Up to rotations, this curve the *Grim Reaper Soliton*. It evolves by translation under the curve shortening flow.

Definition: $C(0)$ is *monotone* if

- $C(0)$ is real analytic.
- $C(0)$ has 4-fold dihedral symmetry.
- $C(0)$ has convex lobes.
- $\kappa_\theta(\theta, 0) > 0$ for $\theta \in (-\alpha(0), \pi/2)$.

The first condition is not much of a restriction because the curve shortening flow instantly turns curves real analytic. Define

$$F(\theta, t) = \frac{\kappa(\theta, t)}{\kappa(\pi/2, t)}. \tag{2}$$

here F is a rescaled version of κ. In this paper we prove the following result.

Theorem 1.1 (Grim Reaper)**.** *Assume that $C(0)$ is monotone. Let $J \subset (0, \pi)$ be an arbitrary closed interval. Let $\epsilon > 0$ be given. For t sufficiently close to T, we have*

$$\sup_{\theta \in J} |F(\theta) - \sin(\theta)| < \epsilon, \qquad \sup_{\theta \in J} |F_\theta(\theta) - \cos(\theta)| < \epsilon.$$

The Grim Reaper Theorem is the analogue of Theorem D in [2]. In [2], S. Angenent also makes versions of the monotonicity assumption. The result implies that a suitably rescaled copy of the arc of $C(t)$ corresponding to $\theta \in (0, \pi)$ converges to the Grim Reaper curve. The arc in question is the one between the two dots in Figure 1. Our proof departs from that in [2] because we are not working with locally convex curves as in [2].

Definition: Let $\psi = \kappa + \kappa_{\theta\theta}$. $C(0)$ is *cincinnous* if

- $C(0)$ is monotone
- $\psi(\pi/2, 0) > 0$.
- $\psi(*, 0)$ vanishes exactly once, and to first order, in the interval $(-\alpha(0), \pi/2)$.

The Lemniscate of Bernoulli is an example of a concinnous figure 8. Also, for any monotone figure 8 curve, ψ attains both positive and negative values. The concinnous condition says that this happens in the simplest possible way.

The *bounding box* of a compact set in the plane is the smallest rectangle, with sides parallel to the coordinate axes, that contains the set. The main goal of the paper is to understand the limit of the sequence $\{L_t(C(t)\}$ where L_t is the positive diagonal matrix such that $L_t(C(t))$ has the square $[-1,1]^2$ for a bounding box. Even though affine transformations do not interact in a nice way with the curve shortening flow, nothing stops us from looking at a solution and applying affine transformations afterwards.

The *bowtie* is the quadrilateral whose vertices are

$$(-1,-1), \quad (1,1), \quad (1,-1), \quad (-1,1)$$

in this cyclic order. It is shaped like this: $\bowtie$. The *Hausdorff distance* between two compact subsets of the plane is the smallest ϵ such that each set is contained in the ϵ-neighborhood of the other. This distance makes the set of compact planar subsets into a metric space. The curves $L_t(C(t))$ move around within a compact set of shapes in this metric space, so it is natural to ask about a limit. Our numerical experiments pointed towards the bowtie.

Figure 1 shows a picture of a numerical simulation of the curve shortening flow. The curve on the left is $L_0(C(0))$ where $C(0)$ is the Leminscate of Bernoulli. The black curve on the right is $L_t(C(t))$ for some later time t. The blue curve on the right is $\Gamma(t) = C(t)/X(t)$, the rescaled version of $C(t)$ whose bounding box has width 2. The black and white dots correspond to where $\theta = 0$ and $\theta = \pi$ respectively. Figure 1 shows some hints of the bowtie forming.

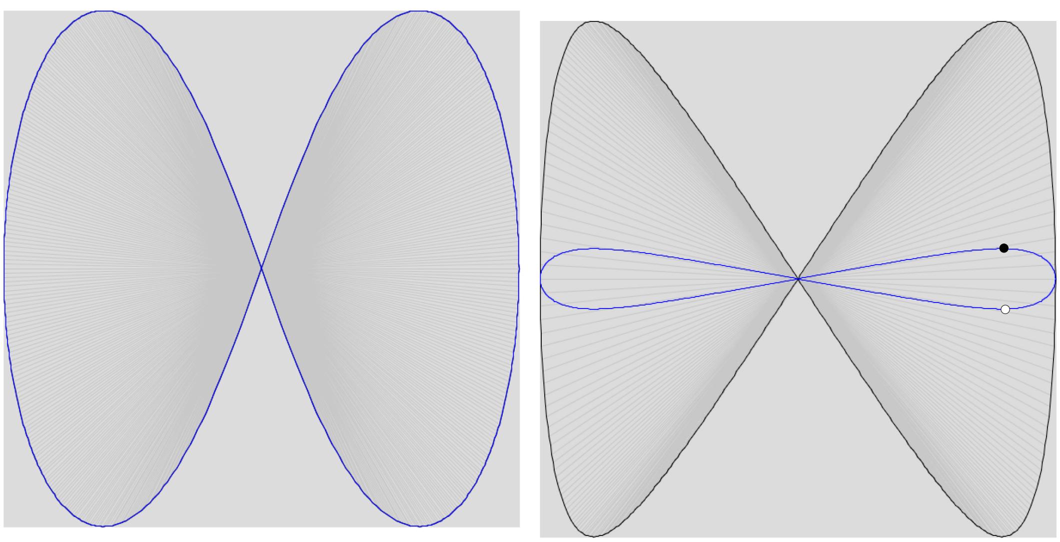

Figure 1: Hints of the bowtie.

Conjecture 1.2 (Bowtie)**.** *Suppose that $C(0)$ is concinnous. As $t \to T$, the sequence $L_t(C(t))$ converges in the Hausdorff metric to the bowtie.*

An earlier version of our paper claimed to have proved this result, but the proof had a gap. Here we will present some partial progress.

Define the *top point* of $L_t(C(t))$ to be the point of the form $(x_t, 1)$. This is the point in the positive quadrant where $L_t(C(t))$ intersects the top of the bounding box. Here is the first of our results.

Theorem 1.3 (Migration)**.** *The top point of $L_t(C(t))$ converges to $(1,1)$ as $t \to T$. Therefore, the two vertical sides of the bowtie are limits of $L_t(C(t))$.*

Our next result shows that the set of possible limiting shapes is, in some sense, near the bowtie.

Theorem 1.4 (Fat Bowtie)**.** *Suppose that $C(0)$ is concinnous. Let $S \subset [-1,1]^2$ denote the set of accumulation points of $L_t(C_t)$ as $t \to \infty$. If $(x,y) \in S$ and $|x| < 1$ then $|x| \leq |y| \leq 2|x| - x^2$.*

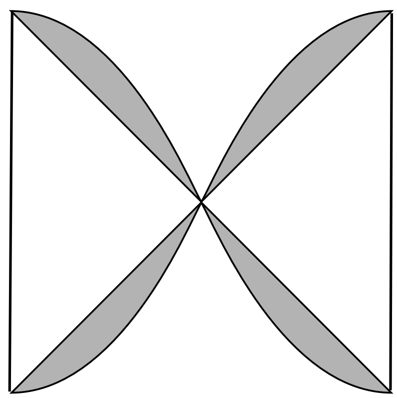

Figure 2: The fat bowtie

Figure 2 shows the region of limit points allowed by the Fat Bowtie Theorem, The bowtie is the portion of the boundary made from straight line segments. The other arcs in the boundary are parts of parabolas.

Here is an outline of the paper. In §2 we prove that the curve shortening flow preserves monotone figure 8 curves, and also preserves concinnous figure 8 curves. In §3 we prove the Grim Reaper Theorem. In §4 we prove the Migration Theorem. In §5 we prove the Fat Bowtie Theorem.

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2 Preservation of the Curvature Conditions

In this chapter we prove that the curve shortening flow preserves monotone figure 8 curves and also concinnous figure 8 curves. That is, if $C(0)$ is monotone (respectively concinnous) then $C(t)$ is monotone (respectively concinnous) for all $t < T$.

2.1 Strictly Parabolic Equations

We begin with a discussion of strictly parabolic equations and two of their basic properties. We follow the notation in [8] and [2].

Let U be an open interval containing $[x_0, x_1]$. We suppose that $u : U \times [0, \tau]$ satisfies the equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u. \tag{3}$$

This equation is called *strictly parabolic* if and only if $a(x,t)$, $b(x,t)$, and $c(x,t)$ are smooth and $a(x,t) > 0$. We assume that u satisfies a strictly parabolic PDE. Here is the well-known *Maximum Principle*.

Theorem 2.1 (The Maximum Principle)**.** *Suppose that $u \neq 0$ on $\{x_0, x_1\} \times [0, \tau]$ and also nonzero on $[x_0, x_1] \times \{0\}$. Then u is nonzero on $[x_0, x_1] \times [0, \tau]$.*

Geometrically we are looking at the behavior of u on a rectangle. If we know that u is nonzero on a certain 3 sides of ∂R then we know u is nonzero on all of R. The side $[x_0, x_1] \times \{0\}$ is the bottom side of R and the side $[x_0, x_1] \times \{\tau\}$ is the top. Here we are picturing time as running vertically and space as running horizontally.

Here is the well-known Sturmian Principle;

Theorem 2.2 (The Sturmian Principle)**.** *Suppose u is nonzero on $\{x_0, x_1\} \times [0, \tau]$. Then the number N_t of times $u(*, t)$ vanishes on (x_0, x_1) is non-increasing with time. Moreover, if $u(*, t)$ vanishes to second order somewhere on (x_0, x_1) then $N_{t'} < N_t$ for all $t' \in (t, \tau]$.*

C. Sturm discovered this principle in 1836. See [15]. The proof of the above version of the Sturmian Principle may be found in [3]. For more references about these theorems, see [8] or [3]. Note that if u, v solve the same strictly parabolic equation then so does $w = u - v$. This yields the following corollary.

Corollary 2.3. *Suppose w is nonzero on $\{x_0, x_1\} \times [0, \tau]$. Then the number N_t of zeroes for $w(*, t)$ on (x_1, x_2) is finite and non-increasing. Moreover, at any time t when $w(*, t)$ vanishes to second order, we have $N_{t'} < N_t$ for all $t' > (t, \tau]$.*

Curvilinear Domains: Corollary 2.3 is too restrictive for one of our purposes. The same principle works when the rectangle in question is replaced by a piecewise analytic quadrilateral $\mathcal{Q}$ with the following two properties:

1. The top and bottom sides are line segments, with the bottom one corresponding to time 0 and the top one corresponding to time τ.

2. The function w does not vanish on the other two sides.

The other two sides play the role of $\{x_0\} \times [0,\tau]$ and $\{x_1\} \times [0,\tau]$. The main issue is that the non-horizontal sides prevent zeros from "leaking in or out".

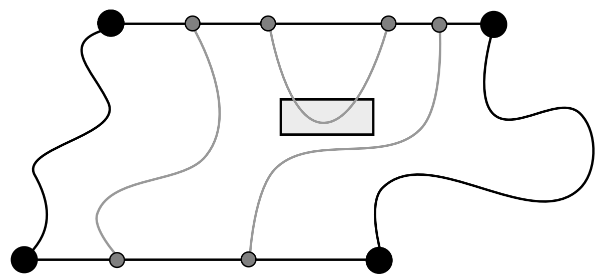

Figure 3: The Curvilinear case

Let us explain why the rectilinear principle implies the curvilinear principle. Suppose we have a situation where w has m zeros on the bottom of $\mathcal{Q}$ and $n > m$ on the top of $\mathcal{Q}$. Let I be the set of times where w has more than m zeros. Let $t = \inf I$. The zeros of w at times converging to t cannot converge to the non-horizontal sides of the domain. Hence at least two of them must coalesce. But then we can find a small rectangle $R \subset \mathcal{Q}$ which surrounds these coalescing points. See the small shaded rectangle in Figure 3. (If more points coalesce, the picture would look more complicated.) This gives a contradiction to the rectilinear principle.

For what it is worth, the curvilinear domains we consider (in the proof of the Sine Lemma below) have only one non-straight side, and this side is the graph of a function $x = g(y)$. See Figure 2.

2.2 Evolution Equations

There are 4 equations we need to consider:

1. The equation for κ.

2. The equation for $u = \kappa_\theta$.

3. The equation for $\psi = \kappa + \kappa_{\theta\theta}$.

4. The equation for the signed curvature as a function of the x-coordinate.

The equation for κ is given in Equation 1. Here it is again:

$$\kappa_t = (\kappa^2)\kappa_{\theta\theta} + (0)\kappa_\theta = (\kappa^2)\kappa \tag{4}$$

The equation for u is

$$u_t = (\kappa^2)u_{\theta\theta} + (2\kappa u)u_\theta + (3\kappa^2)u \tag{5}$$

Let $v = -\kappa_{\theta\theta}$. The equation for v is

$$v_t = \kappa^2 v_{\theta\theta} + 4\kappa u v_\theta - 2\kappa v^2 + 2u^2 v + 3\kappa^2 v - 6\kappa u^2.$$

We observe that $v = \kappa$ also satisfies the same equation:

$$\kappa_t = \kappa^2\kappa_{\theta\theta} + 4\kappa u\kappa_\theta - 2\kappa\kappa^2 + 2u^2\kappa + 3\kappa^2\kappa - 6\kappa u^2 =$$

$$\kappa^2\kappa_{\theta\theta} + (4\kappa u^2 + 2\kappa u^2 - 6\kappa u^2) + (-2\kappa^3 + 3\kappa^3) = \kappa^2\kappa_{\theta\theta} + \kappa^3.$$

Taking the difference of these equations and noting that $\psi = \kappa - v$, we get

$$\psi_t = (\kappa^2)\psi_{\theta\theta} + (-4\kappa u)\psi_\theta + (-2\kappa v + 2u^2 + \kappa^2)\psi. \tag{6}$$

These equations are all valid on the domain

$$\mathcal{D} = \bigcup_{t\in[0,t)} (-\alpha(t), \pi + \alpha(t)) \times \{t\}. \tag{7}$$

Now we get to the final equation. We let k denote the signed curvature, which we think of as a function of x and t. Thus $k(x,y)$ is the curvature at the point $p = (x,y)$ at time t. Note that the domain for x is shrinking to a point. Let $y(x,t)$ be the evolution of the height of the (un-rescaled) curve $C(t)$. The equation for $\mu = k_x$ is

$$\mu_t = (\zeta)\mu_{xx} + (-2y_x y_{xx}\zeta^2)\mu_x + (3k^2)\mu, \qquad \zeta = \frac{1}{1+y_x^2}. \tag{8}$$

One can derive this equation by differentiating the evolution equation for k (as a function of x). The evolution equation for k is worked out in [9] and [12]. The equation for k_x is valid away from places where our curve has vertical tangents. In particular on any time range $[0,t]$ for $t < T$ it is valid on each strand in a fixed neighborhood of the double point.

We have written things so that Equations 4, 5, 6, 8 are clearly strictly parabolic. We never need to know anything about Equations 5, 6, 8 aside from their strict parabolicity, and we only need to consider these 3 equations in this chapter.

2.3 Monotonicity

We first explain why it suffices to consider the case when $k_x(0,0) > 0$. If we are ever treating an initial curve having $k_x(0,0) = 0$, then $k_x(*,0)$ vanishes to order $o_0 \geq 2$ because it is an even function. But then we can apply the Sturmian principle to a small rectangle of the form $[-\epsilon,\epsilon] \times [0,\epsilon]$ to conclude that $k_x(0,\epsilon)$ vanishes to order at most $o_0 - 1$. (In fact we get $o_0 - 2$ because k_x is an even function.) Iterating, we see that for any $\epsilon > 0$ we have $k_x(0,\epsilon) > 0$.

We have $\kappa(\pi/2,t) = 0$ by symmetry. Hence $\kappa_\theta(\pi/2,t)$ vanishes to at least first order for all t. Using the Sturmian Principle as in the previous paragraph, we reduce to the case when $\kappa_\theta(*,0)$ does not vanish to second order at $\pi/2$.

Lemma 2.4. *$k_x(0,t) > 0$.*

Proof. We treat the case of k_x. If the statement about k_x fails, there is some first time t such that $k_x(0,t) = 0$. But $k_x(*,t)$ is analytic for $t \in [0,T)$ and hence its zeros are isolated. We can then apply the Maximum Principle to a rectangle of the form $[-\epsilon,\epsilon] \times [0,t]$ and we get a contradiction. □

Lemma 2.5. *Suppose that $\kappa_\theta(\theta,t) > 0$ for all $\theta \in (0,\pi/2)$ and all $t \leq t_0$. Then $\kappa_\theta(*,t_0)$ vanishes to first order at $\pi/2$.*

Proof. This is an application of the Sturmian Principle to a rectangle of the form $R = [\pi/2-\epsilon, \pi/2+\epsilon] \times [0,t_0]$. The point is that κ_θ does not vanish on the vertical sides of R, by symmetry. □

Lemma 2.6. *If $\kappa_\theta(*,0) > 0$ on $(-\alpha(0),\pi/2)$ then $\kappa_\theta(*,t) > 0$ on $(-\alpha(t),\pi/2)$.*

Proof. Recall that $u = \kappa_\theta$. All that is left to show is that $u(\theta,t) > 0$ on the domain $\mathcal{D}$. Suppose this fails. Let I denote the set of times t' for which $u(*,t')$ vanishes somewhere. Let $t = \inf I$. There are several cases to consider.

Suppose first that $t \in I$. Then there is some $(\theta,t) \in \mathcal{D}$ such that $u(\theta,t) = 0$ but $u(*,t') > 0$ for all $t' \in [0,t)$. In this case we get a contradiction by applying the Maximum principle to u on a rectangle

$$[\theta-\epsilon,\theta+\epsilon] \times [t-\epsilon,t].$$

For sufficiently small ϵ this rectangle belongs to $\mathcal{D}$. Since $u(*,t)$ is analytic we can further choose ϵ so that $u(\theta \pm \epsilon,t) > 0$. We now contradict the Maximum Principle. Hence $t \notin I$.

Let (θ_n,t_n) be a sequence of points in $\mathcal{D}$ such that $u(\theta_n,t_n) = 0$ and $t_n \to t$. Since $t \notin I$, we must have (after using symmetry and passing to a subsequence) either $\theta_n \to -\alpha(t)$ or $\theta_n \to \pi/2$. Intuitively, what we are saying is that the zeros must leak in from the left or the right boundary component. We consider the cases in turn.

- Suppose $\theta_n \to -\alpha(t)$. By the Chain rule, $k_x(x_n, t_n) = 0$ for a sequence $x_n \to 0$. But then $k_x(0,t) = 0$ by continuity. This contradicts the fact that $k_x(0,t) > 0$.

- Suppose $\theta_n \to \pi/2$. Since we are now in the interior of the domain $\mathcal{D}$ and u is a smooth function, we have

$$\kappa_{\theta\theta}(\pi/2, t) = \lim_{n\to\infty} \frac{u(\pi/2, t_n) - u(\theta_n, t_n)}{\pi/2 - \theta_n} = 0.$$

  This means that $\kappa_\theta(*, t)$ vanishes to second order, contradicting Lemma 2.5.

This completes the proof. □

These lemmas combine to show that curve shortening preserves the monotonicity property.

2.4 Two Integral Formulas

To help analyze the concinnity condition we establish an integral formula for the curvature This formula is essentially the same as the one that appears in the proof of Lemma 8.2 in [2]. Even though we won't need it until later chapters, we also deduce a second integral formula.

Lemma 2.7.

$$\kappa(\theta, t) = \kappa(\pi/2, t)\sin(\theta) + \int_\theta^{\pi/2} \sin(\phi - \theta)\big(\kappa_{\theta\theta}(\phi, t) + \kappa(\phi, t)\big)d\phi. \tag{9}$$

Proof. We set $c(\cdot) = \cos(\cdot)$ and $s(\cdot) = \sin(\cdot)$. We have:

$$\frac{\partial}{\partial\theta}\bigg(s(\theta)\kappa_\theta(\theta, t) - c(\theta)\kappa(\theta, t)\bigg) = s(\theta)(\kappa_{\theta\theta}(\theta, t) + \kappa(\theta, t)).$$

$$\frac{\partial}{\partial\theta}\bigg(c(\theta)\kappa_\theta(\theta, t) + s(\theta)\kappa(\theta, t)\bigg) = c(\theta)(\kappa_{\theta\theta}(\theta, t) + \kappa(\theta, t)).$$

Setting $\psi = \pi/2$ and integrating these identities from θ to ψ, we compute:

$$\int_\theta^\psi s(\phi - \theta)\big(\kappa_{\theta\theta}(\phi, t) + \kappa(\phi, t)\big)d\phi =$$

$$c(\theta)\int_\theta^\psi s(\phi)\big(\kappa_{\theta\theta}(\phi,t)+\kappa(\phi,t)\big)d\phi - s(\theta)\int_\theta^\psi c(\phi)\big(\kappa_{\theta\theta}(\phi,t)+\kappa(\phi,t)\big)d\phi =$$
$$c(\theta)s(\psi)\kappa_\theta(\psi,t) - c(\theta)c(\psi)\kappa(\psi,t) - c(\theta)s(\theta)\kappa_\theta(\theta,t) + c(\theta)c(\theta)\kappa(\theta,t)$$
$$-s(\theta)c(\psi)\kappa_\theta(\psi,t) - s(\theta)s(\psi)\kappa(\psi,t) + s(\theta)c(\theta)\kappa_\theta(\theta,t) + s(\theta)s(\theta)\kappa(\theta,t) =$$
$$\kappa(\theta,t) - s(\theta)\kappa(\theta,t).$$

We repeatedly use $\cos(\psi)=0$ and $\sin(\psi)=1$. The last equality uses the fact that $\kappa_\theta(\psi,t)=0$. $\square$

Here is our second integral formula.

Lemma 2.8.

$$\cos(\theta)\kappa(\theta,t) - \sin(\theta)\kappa_\theta(\theta,t) = \int_\theta^{\pi/2} \sin(\phi)(\kappa_{\theta\theta}(\phi,t)+\kappa(\phi,t))d\phi. \tag{10}$$

Proof. Differentiating Equation 9 with respect to θ we have

$$\kappa_\theta(\theta,t) = \kappa(\pi/2,t)\cos(\theta) - \int_\theta^{\pi/2} \cos(\phi-\theta)\big(\kappa_{\theta\theta}(\phi,t)+\kappa(\phi,t)\big)d\phi. \tag{11}$$

Now we multiply Equation 9 by $\cos(\theta)$ and Equation 11 by $\sin(\theta)$ and subtract. This tells us that the left hand side of Equation 10 equals

$$\int_\theta^{\pi/2} \Sigma(\theta,\phi)\big(\kappa_{\theta\theta}(\phi,t)+\kappa(\phi,t)\big)d\phi,$$

where

$$\Sigma(\theta,\phi) = \sin(\phi-\theta)\cos(\theta) + \cos(\phi-\theta)\sin(\theta) = \sin(\phi).$$

The last equality is the angle addition formula for the sine function. This gives us the right side of Equation 10. $\square$

2.5 Concinnity

In this section we show that if $C(0)$ is concinnous then so is $C(t)$ for all $t \in (0,T)$.

Lemma 2.9. $\psi(\pi/2) > 0$ *for all* $t \in (0,T]$.

Proof. Suppose this is false. Let I be the set of times t' where $\psi(\pi/2, t') = 0$. Since $\pi/2$ is in the interior of the domain where $\psi(*, t)$ is defined, the set I contains its infimim: There is some t such that $\psi(\pi/2, t) = 0$ but $\psi(\pi/2, t') > 0$ for all $t' < t$.

This situation contradicts the Sturmian principle unless we can find a sequence $t_n \to t$, from below, with the property that $\theta_n \to \pi/2$. Here θ_n is the point such that $\psi(\theta_n, t_n) = 0$. But then, taking a limit, we see that $\psi(*, t) \leq 0$ on all of $[0, \pi/2]$. This contradicts our first integral formula:

$$\kappa(0, t) = \int_0^{\pi/2} \sin(\phi)\psi(\phi, t)d\phi > 0$$

This completes the proof. □

Lemma 2.10. *$\psi(*, t)$ vanishes at least once in $(-\alpha(t), \pi/2)$.*

Proof. It suffices to show that ψ takes both signs on this interval. Here is the argument for positivity: Our integral formula

$$\kappa(0, t) = \int_0^{\pi/2} \sin(\phi)\psi(\phi, t)d\phi > 0$$

shows that $\psi(*, t) > 0$ somewhere on $(0, \pi/2)$.

Here is the argument for negativity: Let $\theta', \theta < 0$ be any two parameters, and let s', s be the corresponding arc-length parameters. We normalize so that $s' = 0$ corresponds to $\theta = -\alpha(t)$. We have

$$\int_{s'}^{s} \frac{\kappa_s}{\kappa} ds = \int_{s'}^{s} \frac{d}{ds} \log(\kappa_s) ds = \log \frac{\kappa(s, t)}{\kappa(s', t)}.$$

In the last equation we are thinking of κ as a function of arc length. Letting $s' \to 0$ we can make the right hand side of this equation as large as we like. Hence $\kappa_\theta = \kappa_s/\kappa$ is unboundely positive on the angle interval

$$I_\theta = (-\alpha(t), \theta).$$

This can only happen if $\kappa_{\theta\theta}$ is unboundedly negative on I_θ. This is true for any $\theta > -\alpha(t)$. We now pick θ so small that $\kappa(*, t) < 1$ on I_θ. But now we have shown that ψ is unboundedly negative on I_θ. □

Lemma 2.11. *There does not exist a time t such that $C(t)$ is not concinnous but $C(t')$ is concinnous for all $t' < t$.*

Proof. If t' is sufficiently close to t we can choose a rectangle R, contained entirely in the domain for the flow, with the following properties.

- ψ is defined in a neighborhood of R.
- The top edge of R corresponds to t.
- ψ vanishes at least twice (counted with multiplicity) on the top edge of R.
- The bottom edge corresponds to t'.
- ψ does not vanish on the vertical edges of R.

Since ψ vanishes at most once on the bottom edge of R we contradict the Sturmian Principle. □

If $C(t)$ is not always concinnous then, given the previous lemma, there is a sequence of times $t_n \to t$ (from above) such that $\psi(*, t_n)$ vanishes at least twice counting multiplicity. If there are two distinct values $\theta_{n,1}$ and $\theta_{n,2}$ where $\psi(*, t_n)$ vanishes, then at least one of the sequence $\{\theta_{n,1}\}$ or $\{\theta_{n,2}\}$ has a subsequence which converges to $-\alpha(t)$. Otherwise we could take a limit and see that $\psi(*, t)$ vanishes at least twice with multiplicity. The same argument works if there is just one value θ_n where $\psi(*, t_n)$ vanishes (with multiplicity greater than 1).

We extract a subsequence $\{\theta_n\}$ converging to $-\alpha(t)$. Again, $\psi(\theta_n, t_n) = 0$. To analyze this situation we treat curvature as a function of arc-length. We let $\overline{k}(s, t)$ denote the curvature at the arc length parameter s and at time t. We normalize so that $s = 0$ correponds to the double point.

We have the change of variables formula derived in [9].

$$\overline{k}_s = \kappa\kappa_\theta, \qquad \overline{k}_{ss} = \kappa\kappa_\theta^2 + \kappa^2\kappa_{\theta\theta}. \tag{12}$$

Let s_n be the arc length parameter corresponding to θ_n. Note that $\overline{k}_{ss}(0, t) = 0$ because k is an odd function. Since $\overline{k}_{ss}(0, t) = 0$, the second equation in Equation 12 combines with the fact that

$$\kappa_{\theta\theta}(\theta_n, t_n) = -\kappa(\theta_n, t_n)$$

to imply:

$$\lim_{n\to\infty} \left(\kappa(\theta_n, t_n)\kappa_\theta^2(\theta_n, t_n) - \kappa^3(\theta_n, t_n) \right) = 0$$

At the same time,

$$\lim_{n\to\infty} \kappa^3(\theta_n, t_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} \kappa(\theta_n, t_n)\kappa_\theta^2(\theta_n, t_n) = 0 \tag{13}$$

By definition and by Equation 12 we have

$$\kappa_\theta(\theta_n, t_n) = \frac{\overline{k}_s(s_n, t_n)}{k(s_n, t_n)}. \tag{14}$$

Combining this information with Equation 13 we have

$$\lim_{n\to\infty} \frac{\overline{k}_s^2(s_n, t_n)}{\overline{k}(s_n, t_n)} = 0. \tag{15}$$

But we also have

$$\lim_{n\to\infty} \overline{k}(s_n, t_n) = \overline{k}(0, t) = 0, \qquad \lim_{n\to\infty} \overline{k}_s(s_n, t_n) = \overline{k}_s(0, t) = k_x(0, t) \times \frac{dx}{ds} > 0. \tag{16}$$

The last equality comes from Lemma 2.4. Equation 16 implies that the limit in Equation 15 is actually ∞ rather than 0, and we have a contradiction.

3 The Grim Reaper Theorem

In this chapter we prove the Grim Reaper Theorem. Recall that $F(\theta, t) = \kappa(\theta, t)/\kappa(\pi/2, t)$. Here is the theorem again:

Theorem 3.1 (Grim Reaper). *Assume that $C(0)$ is monotone. Let $J \subset (0, \pi)$ be an arbitrary closed interval. Let $\epsilon > 0$ be given. For t sufficiently close to T, we have*

$$\sup_{\theta \in J} |F(\theta) - \sin(\theta)| < \epsilon, \qquad \sup_{\theta \in J} |F_\theta(\theta) - \cos(\theta)| < \epsilon.$$

3.1 The Evolution of the Bounding Box

Let $[-X(t), X(t)] \times [-Y(t), Y(t)]$ be the bounding box of $C(t)$.

Lemma 3.2 (Bounding Box). $\lim_{t\to T} Y(t)/X(t) = 0$.

Proof. The perimeter of $C(t)$ and the area of the region bounded by $C(t)$ are respectively within a factor of 2 of the perimeter and area of the bounding box of $C(t)$. Thus, Grayson's isoperimetric result tells us that the aspect ratio of the bounding box tends to 0. This means that either $Y(t)/X(t) \to 0$ or $Y(t)/X(t) \to \infty$ as $t \to T$.

We establish in §2.3 that $C(t)$ is concinnous for all $t \in (0,T)$. We assume that result here. Since $X_t(t) = -\kappa(\pi/2,t)$ and $Y_t(t) = -\kappa(0,t)$, we have

$$Y(t) = \int_t^T \kappa(0,u)\ du < \int_t^T \kappa(\pi/2,u)\ du = X(t).$$

This rules out the second option above. □

3.2 Some Asymptotics

The lemmas in this section rely on Lemma 3.2 and the Tait-Kneser Theorem.

Theorem 3.3 (Tait-Kneser)**.** *Suppose γ is a curve of strictly monotone increasing curvature. Then the osculating disks of γ are strictly nested. The largest one is at the initial endpoint and the smallest one is at the final endpoint.*

Here are the applications.

Lemma 3.4. $\lim_{t\to\infty} \kappa(\theta,t) = \infty$ *for any* $\theta \in (0,\pi/2]$.

Proof. Let $\Gamma(t) = C(t)/X(t)$. This is a rescaled version of $C(t)$ whose bounding box has width 2. The height of the bounding box tends to 0 by Lemma 3.2. Let $K(\theta,t) = X(t)\kappa(\theta,t)$ be the curvature of $\Gamma(t)$ at the point where the tangent angle is θ. Since $\lim_{t\to T} X(t) = 0$, it suffices to prove that there is some constant $a = a_\theta > 0$ such that $K(\theta,t) > a$ for all t sufficiently close to T.

Suppose that this result is false. Let $D(\theta,t)$ be the osculating disk to $\Gamma(t)$ at θ. Let H be the horizontal line connecting the point $\Gamma(\theta,t)$ to the y-axis. The circle ∂D makes an angle of θ with H at the right endpoint of H. By the Tait-Kneser Theorem, $\partial D(\theta,t)$ crosses H at a second point. The second crossing angle is also θ. Since the length of H is at most 1 unit, we see from trigonometry that the radius of $D(\theta,t)$ is at most $1/(2\sin(\theta))$. Hence the curvature of $\Gamma(t)$ is at least $2\sin(\theta)$. This is to say that

$$\kappa(\theta,t)X(t) \geq 2\sin(\theta). \tag{17}$$

Since $X(t) \to 0$ as $t \to T$ we see that $\kappa(\theta,t) \to \infty$. □

Lemma 3.5. $\lim_{t\to\infty} \alpha(t) = 0$.

Proof. Let $\Gamma(t)$ be as in the previous lemma. Suppose that there is a sequence of times $t_n \to T$ such that $\alpha(t_n) > \delta > 0$ for some constant δ. Let L be the line through the origin which makes an angle of $\delta/2$ with the x-axis. By Lemma 3.2, the height of the bounding box $\beta(t)$ for $\Gamma(t)$ tends to 0 as $t \to T$. Hence, L hits the top of $\beta(t)$ at a point whose distance to the origin tends to 0 as $t \to T$.

By construction $\Gamma(t_n)$ starts out from the origin lying to the left of L. Since $\Gamma(t_n) \subset \beta(t_n)$, we see that $\Gamma(t_n)$ crosses L at some point p_n such that $\|p_n\| \to 0$. The total variation of the tangent angle of $\Gamma(t)$ along the arc connecting $(0,0)$ to p_n is, by convexity, at least $\delta/2$. Since the length of this arc tends to 0, some point q_n on this arc has curvature at least 4. By construction $\|q_n\| \to 0$.

By the Tait-Kneser Theorem the arc of $\Gamma(t)$ connecting q_n to $(1,0)$ is trapped in a disk of radius $1/4$ which contains q_n in its boundary. This is a contradiction. $\square$

3.3 Counting Zeros

Our final lemma has nothing to do with the flow. A very similar principle is used in [2]. Let $J \subset \mathbf{R}$ be some interval. Call a function $g : J \to \mathbf{R}$ *small* if

$$\sup_J g^2 + (g')^2 < 1. \tag{18}$$

Call J *small* if it has length at most π. Every small interval is contained in a closed interval of length π. Closed intervals of length π count as being small.

Lemma 3.6. *If g is a small function and J is a small interval then the difference $w(x) = g(x) - \sin(x)$ vanishes at most twice on J, counting multiplicity.*

Proof. Let $f(x) = \sin(x)$. We note the crucial property that

$$f^2 + (f')^2 = 1 > g^2 + (g')^2.$$

Let F and G respectively denote the graphs of F and G. These graphs must be transverse wherever they intersect. Otherwise we would have $g^2 + (g')^2 = 1$ at an intersection point. This is impossible. We show that $f = g$ at most twice. Given the transversality just mentioned, this is equivalent to the statement that $w = g - f$ vanishes at most twice on J, counting multiplicity.

As usual in calculus, say that $x \in J$ is an *extreme point* if $f'(x) = 0$. The only way that J can contain two extreme points is if J has length π, and the endpoints are the two extreme points, and $|f| = 1$ at these endpoints. In this case $f \neq g$ at the endpoints because $|g| < 1$. So, even in this case, we can replace J by a smaller interval which contains all the points where $f = g$. Thus, we can assume without loss of generality that J contains at most one extreme point.

Suppose first that J has no extreme points. Then f is either monotone increasing on J or monotone decreasing. Consider the case when f is monotone increasing.

Suppose it happens that there are two consecutive points $x_1, x_2 \in J$ where f and g agree. The portion of G between $(x_1, g(x_1))$ and $(x_2, g(x_2))$ either lies above F or below. In the first case we have $g'(x_1) > f'(x_1)$, which is a contradiction. In the second case we have $g'(x_2) > f'(x_2)$ and we have the same contradiction. Hence $f(x) = g(x)$ for at most one point $x \in J$. The same argument works when f is monotone decreasing on J.

Now consider the case when J has exactly one extreme point. In this case we can write $J = J_1 \cup J_2$ where f is monotone on each J_i. In this case, the same argument above, applied to each of these sub-intervals, shows that they each have at most one point where $f = g$. Hence J has at most 2 such points. □

3.4 The Sine Lemma

Here is the crucial step in the proof of the Grim Reaper Theorem. This section is devoted to proving the following result.

Lemma 3.7 (Sine)**.** *Let J be any closed interval contained in $(0, \pi)$. Let $\epsilon > 0$ are given. If t is sufficiently close to T then*

$$\left|\frac{\kappa_\theta(\theta, t)}{\kappa(\theta, t)} - \frac{\cos(\theta)}{\sin(\theta)}\right| < \epsilon,$$

for all $\theta \in J$.

We will assume for the sake of contradiction that there is a sequence of times $t_n \to T$ and a sequence $\{\theta_n\} \in J$ such that

$$\left|\frac{\kappa_\theta(\theta_n, t_n)}{\kappa(\theta_n, t_n)} - \frac{\cos(\theta_n)}{\sin(\theta_n)}\right| > \epsilon. \tag{19}$$

Passing to a subsequence we can assume that $\theta_n \to \theta_0 \in J$. By compactness of J we can choose a constant We choose a constant $\Sigma = \Sigma(J, \epsilon) > 0$ so that

$$\left|\frac{\cos(\phi + \theta_n)}{\sin(\phi + \theta_n)} - \frac{\cos(\theta_n)}{\sin(\theta_n)}\right| > \epsilon \qquad \Longrightarrow \qquad |\phi| > \Sigma, \tag{20}$$

as long as $\phi + \theta_n \in (0, \pi)$.

Call the non-horizontal sides of our domains the *sidewalls*. Thanks to Lemma 3.5 we can omit the initial portion of our evolution and arrange that

$$\sup_{t \in [0,T)} \alpha(t) < 10^{-100}\Sigma. \tag{21}$$

We are making the horizontal displacement of the sidewalls of $\mathcal{D}$ extremely small in comparison to the other relevant quantities that arise below. We don't need the factor of 10^{-100}; we add it for emphasis.

Let

$$C = \sup_{\theta \in [0,\pi]} \kappa^2(\theta,0) + \kappa_\theta^2(\theta,0), \qquad B_n = \kappa^2(\theta_n,t_n) + \kappa_\theta^2(\theta_n,t_n). \tag{22}$$

By Lemma 3.4 there is some n such that $B_n > C$. Our notivation for taking $B_n > C$ is the following corollary of Lemma 3.6.

Corollary 3.8. *Suppose*

$$\sup_{\theta \in J} \kappa^2(\theta,0) + \kappa_\theta^2(\theta,0) \leq C.$$

Let $S(\theta) = \sqrt{B}\sin(\phi+\theta)$ *for any value* ϕ. *If* $B > C$ *then* $w(*) = \kappa(*,0) - S(*)$ *vanishes at most twice on* J, *counting multiplicity.*

Proof. This follows from Lemma 3.6 by symmetry and scaling. □

We fix n for which $B_n > C$. We set $B = B_n$ and $t = t_n$. There is some angle ϕ such that

$$\frac{\kappa_\theta(\theta_n,t)}{\kappa(\theta_n,t)} = \frac{\cos(\phi+\theta_n)}{\sin(\phi+\theta_n)}.$$

For this choice of ϕ we have

$$S(\theta_n) = \sqrt{B}\sin(\phi+\theta_n) = \kappa(\theta_n,t), \qquad S_\theta(\theta_n) = \sqrt{B}\cos(\phi+\theta_n) = \kappa_\theta(\theta_n,t). \tag{23}$$

Our function S determines a unique interval I of length π such that $S > 0$ on the interior of I and $\theta_n \in I$. Note also that $S = 0$ on ∂I. Let $\Omega = I \times [0,t]$. This is exactly the domain considered in [2], but now our proof departs from [2].

Lemma 3.9. *One sidewall of* Ω *is disjoint from the closure of* $\mathcal{D}$ *and the other sidewall of* Ω *lies in* $\mathcal{D}$.

Proof. The properties of S imply the following:

$$\left|\frac{\cos(\phi+\theta_0)}{\sin(\phi+\theta_0)} - \frac{\cos(\theta_0)}{\sin(\theta_0)}\right| = \left|\frac{\kappa_\theta(\theta_0,t)}{\kappa(\theta_0,t)} - \frac{\cos(\theta_0)}{\sin(\theta_0)}\right| > \epsilon. \tag{24}$$

By equation 20, we have $|\phi| > \Sigma$.

If we had $\phi = 0$ we would have $I = [0, \pi]$. As it is, we have $|\phi| > \Sigma$. This shifts I and Ω by at least Σ to the left or to the right. Given our bound on the horizontal displacement of the sidewalls of $\mathcal{D}$, this shift causes one sidewall or the other to stick out completely. See Figure 2 below. At least one point of I lies in $(0, \pi)$ and the total width of I is π. Hence I cannot both contain points less than 0 and greater than π. This means that the other sidewall lies inside $\mathcal{D}$. $\square$

We now create a new domain $\mathcal{Q}$ by intersecting Ω with $\mathcal{D}$ and pushing in the curvilinear sidewall a bit. We treat the case when Ω sticks out on the left. The other case is essentially the same.

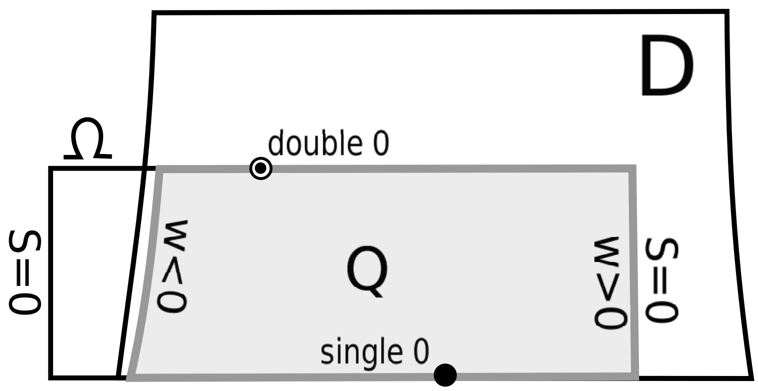

Figure 2: The new domain $\mathcal{Q}$, shaded.

Define

$$w(\theta, t) = \kappa(\theta, t) - S(\theta). \tag{25}$$

The function S is a stationary solution to Equation 1, meaning that $S_t = 0$. This means that w is exactly the sort of difference of solutions to which the Sturmian Principle applies. Let us examine the behavior of w on the boundary of $\mathcal{Q}$.

Left: Since κ limits to 0 on the sidewalls of $\mathcal{D}$ and $S > 0$ on the left sidewall of $\mathcal{D}$, we can by compactness make the perturbation small enough so that $w < 0$ on the left sidewall of $\mathcal{Q}$.

Right: The right sidewall of $\mathcal{Q}$ lies in $\mathcal{D}$. Since $S = 0$ on the right sidewall of $\mathcal{Q}$ and $\kappa > 0$ everywhere in $\mathcal{D}$, we have $w > 0$ on the right sidewall of $\mathcal{Q}$.

Bottom: Applying Corollary 3.8 to the bottom side J of $\mathcal{Q}$, we see that $w(*, 0)$ vanishes at most twice on J counting multiplicity. Since w has opposite signs on the sidewalls of $\mathcal{Q}$ the number of zeros of w on J is odd, counting multiplicity. Since this number is at most 2, it must be exactly 1. In short, w vanishes exactly once on the bottom side of $\mathcal{Q}$, counting multiplicity.

Top: On the top side J' of $\mathcal{Q}$ we have arranged that w and w_θ vanish at (θ_0, t). This means that w vanishes at least twice, counting multiplicity, on J'. We have shown this double point in Figure 2. Since w has opposite signs on the sidewalls of $\mathcal{Q}$ the number of zeros of w on J' is odd, counting multiplicity. Since this number is at least 2 it is actually at least 3. In short, w vanishes at least 3 times on the top side of $\mathcal{Q}$ counting multiplicity.

The above properties violate the Sturmian Principle for (Equation 1, $\mathcal{Q}$, w). This completes the proof of the Sine Lemma.

3.5 The End of the Proof

In this section we prove the Grim Reaper Theorem.

Corollary 3.10. *Let $\epsilon > 0$ be given and let $J \subset (0, \pi)$ be any closed interval. We have*

$$\sup_{\theta \in J} \left| \frac{F_\theta(\theta, t)}{F(\theta, t)} - \frac{\cos(\theta)}{\sin(\theta)} \right| < \epsilon,$$

for t sufficiently close to T.

Proof. We can replace κ by F because for each time these functions are constant multiples of each other. $\square$

Consider the new function

$$G(\theta, t) = \frac{F(\theta, t)}{\sin(\theta)}. \tag{26}$$

Using Lemma 3.10 we have the following result:

$$|G_\theta| = \frac{|F_\theta(\theta, t)\sin(\theta) - F(\theta, t)\cos(\theta)|}{\sin^2(\theta)} < \frac{\epsilon F(\theta, t)\sin(\theta)}{\sin^2(\theta)} = \epsilon G, \tag{27}$$

This holds for all $\theta \in J$ provided that we take t sufficiently close to T. The last calculation shows that the logarithmic derivative G_θ / G is nearly 0 on J. Hence G is nearly constant on J. But $G(\pi/2, t) = 1$. Hence G is nearly 1 on J. This proves that $F(\theta, t)$ converges uniformly to $\sin(\theta)$ for $t \in J$. But this combines with Corollary 3.10 to show that $F_\theta(\theta, t)$ converges uniformly to $\cos(\theta)$ for $t \in J$. This completes the proof of the Grim Reaper Theorem.

4 The Migration Theorem

In this chapter we prove the Migration Theorem. Here it is again.

Theorem 4.1 (Migration)**.** *The top point of $L_t(C(t))$ converges to $(1,1)$ as $t \to T$. Therefore, the two vertical sides of the bowtie are limits of $L_t(C(t))$.*

4.1 The Geometric Part of Argument

Recall that the bounding box of $C(t)$ is $[-X(t), X(t)] \times [-Y(t), Y(t)]$. Let $(x^*(t), Y(t))$ denote the topmost point on the right lobe of $C(t)$. The Migration Theorem is equivalent to the statement that

$$\lim_{t \to T} \frac{x^*(t)}{X(t)} = 1. \tag{28}$$

In this section we derive Equation 28 from the following asymptotic formula.

$$\lim_{t \to T} Y(t)\kappa(\pi/2, t) = \pi/2. \tag{29}$$

Following this section, we prove Equation 29.

Let $H(t) = Y(t)/X(t)$. Let $\Gamma(t) = C(t)/X(t)$.

Lemma 4.2. *Let $\epsilon > 0$ be given. If $\delta > 0$ is sufficiently small, the arc connecting $\Gamma(\pi/2, t)$ to $\Gamma(\delta, t)$ has vertical displacement at least $H(t)(1-\epsilon)$ provided that t is sufficiently close to T.*

Proof. We are going to rescale several times. The Grim Reaper Curve $G = G(\theta)$ has the property that its maximum curvature is 1 at the point $G(\pi/2)$ and its total width is π. In other words, if we atart at the point of maximum curvature of the Grim Reaper and travel outward, we gain a height of $\pi/2$. This means that if $\delta > 0$ is sufficiently small, then the portion of G connecting $G(\pi/2)$ to $G(\delta)$ has a vertical displacement of at least $\pi/2(1-\epsilon)$.

Let $H(t)$ be the curve we get by rescaling $C(t)$ by a factor of $(\pi/2)Y(t)$. By Equation 29 and the Grim Reaper Theorem together, this curve converges uniformly to the Grim Reaper (modulo translations) on every interval $[\delta, \pi - \delta]$. This means that $H(t)$ rises up to $\pi/2(1-\epsilon)$ before reaching $H(\delta)$, provided that δ is sufficiently small. This is equivalent to saying that the portion of $C(t)$ connecting $C(\pi/2)$ to $C(\delta)$ has vertical displacement at least $Y(t)(1-\epsilon)$ provided that δ is sufficiently small. This result is equivalent to the claim of the lemma, by scaling. $\square$

Now we suppose that Equation 28 is false and we derive a contradiction. We can find a sequence $t_n \to T$ such that the distance from $\Gamma(0, t_n)$ to the vertical line $x = 1$ is at least some fixed $\eta > 0$. At the same time, the distance from $\Gamma(\pi/2, t_n)$ to $\Gamma(\delta, t_n)$ is less than $\eta/2$ once n is sufficiently large. This is a consequence of the Tait-Kneser Theorem and the fact that the curvature of $\Gamma(t_n)$ tends uniformly to ∞ on the interval $[\delta, \pi/2]$. We conclude from this that the arc of $\Gamma(t_n)$ connecting $\Gamma(0, t_n)$ to $\Gamma(\delta, t_n)$ has a horizontal displacement of at least $\delta/2$ and a vertical displacement of at most $H(t_n)\epsilon$.

Let $D(t_n)$ be the disk which osculates $\Gamma(t_n)$ at $\Gamma(0, t_n)$. Let $\partial_+ D(t_n)$ be the portion of $\partial D(t_n)$ which connects $\Gamma(0, t_n)$ to the vertical line above $\Gamma(\delta, t_n)$. Let $\partial_- D(t_n)$ be the portion of $\partial D(t_n)$ which connects $\Gamma(0, t_n)$ to the y-axis – i.e., the vertical line through the double point. The horizontal displacement of $\partial_+ D(t_n)$ is at least $\eta/2$. The horizontal displacement of $\partial_- D(t_n)$ is at most 1. This is true independent of n. Hence, the vertical displacement of $\partial_+ D(t_n)$ is at most K times the vertical displacement of $\partial_- D(t_n)$ for some constant K that does not depend on n. Let us write this as

$$v_- < K v_+ \tag{30}$$

By the Tait-Kneser Theorem, the curve connecting $\Gamma(0, t_n)$ to $\Gamma(\delta, t_n)$ remains below $\partial_+ D(t_n)$. Hence

$$v_+ < H(t_n)\epsilon.$$

We choose δ so small that $\epsilon K < 1$. This means that

$$v_- < H(t_n),$$

By the Tait-Kneser Theorem, the portion of $\Gamma(t_n)$ connecting $\Gamma(0, t_n)$ to the origin lies above $\partial_- D(t_n)$. But this forces $v_- \geq H(t_n)$. We have a contradiction. This establishes Equation 28 modulo the proof of Equation 29. The rest of the chapter is devoted to proving Equation 29.

4.2 The Analytic Part of the Argument

It remains to establish Equation 29. This formula essentially follows from the Grim Reaper Theorem, but we need an extra step to control a potentially unbounded integral.

Lemma 4.3.

$$Y(t)\kappa(\pi/2, t) = \int_0^{\pi/2} \frac{\sin(\phi)}{F(\phi, t)} d\phi. \tag{31}$$

Proof. Let s_0 and s_1 respectively denote the arc-length parameters that correspond to $\theta_0 = 0$ and $\theta_1 = \pi/2$. On the level of 1-forms:

$$dy = -ds\sin\theta, \qquad \kappa(\theta,t)ds = d\theta.$$

(The minus sign appears because y decreases as s increases.)

$$Y(t) = \int_0^{Y(t)} dy = -\int_{s_1}^{s_0} \sin(\theta)ds = \int_{s_0}^{s_1} \sin(\theta)ds = \int_0^{\pi/2} \frac{\sin(\theta)}{\kappa(\theta,t)}d\theta. \tag{32}$$

Multiplying through by $\kappa(\pi/2,t)$, we get Equation 31. $\square$

By the Grim Reaper Theorem, the integrand in Equation 31 is converging to 1 at every point of $(0,\pi/2)$. This would suggest that the total integral is $\pi/2$. However, we don't have much control over how $F(\theta,t) \to 0$ as $\theta \to 0$. This is where the concinnity condition comes in. In the next section we will prove the following estimate:

Lemma 4.4. *For any $\delta \in (0,\pi/2)$ we have*

$$\int_0^\delta \frac{\sin(\theta)}{\kappa(\theta,t)}d\phi < \frac{\delta^2}{\kappa(\delta,t)}.$$

Multiplying through by $\kappa(\pi/2,\theta)$ we have

$$\int_0^\delta \frac{\sin(\theta)}{F(\theta,t)}d\phi < \frac{\delta^2}{F(\delta,t)} < 2\delta. \tag{33}$$

The last inequality holds once t is sufficiently close to T. The "closeness" needed depends on the choice of δ.

For each $\delta \in (0,\pi/2)$ the Grim Reaper Theorem gives

$$\lim_{t\to T} \Theta_\delta = \pi/2 - \delta, \qquad \Theta_\delta = \int_\delta^{\pi/2} \frac{\sin\phi}{F(\phi,t)}d\phi. \tag{34}$$

Hence, for t sufficiently close to T, we have

$$\pi/2 - 2\delta < \Theta_\delta < Y(t)\kappa(\pi/2,t) = \Theta_\delta + \int_0^\delta \frac{\sin(\phi)}{F(\phi,t)}d\phi < \pi/2 + 2\delta. \tag{35}$$

Since δ is arbitrary, we get Equation 29.

4.3 The Crucial Estimate

In this section we finish the proof of the Migration Theorem by establishing Lemma 4.4. First we prove two preliminary results. Our concinnity assumption is that $\psi(*, t)$ vanishes exactly once, at some value $\overline{\theta} \leq \pi/2$. In view of Lemma 2.9, this means that $\psi < 0$, then ψ vanishes, then $\psi > 0$. Introduce the quantity

$$I(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \sin(\phi)\psi(\phi, t) d\phi. \tag{36}$$

Lemma 4.5. *$I(\theta, \pi/2) > 0$ for all $\theta \in [0, \pi/2)$.*

Proof. This is obvious if $\overline{\theta} \leq 0$. So, we may assume that $\overline{\theta} \in (0, \pi/2]$. The result is again obvious if $\theta = 0$ and if $\theta \geq \overline{\theta}$. Henceforth take $\theta \in (0, \overline{\theta})$. Note that $-I(0, \theta) \geq 0$. We have

$$I(\theta, \pi/2) = I(0, \pi/2) - I(0, \theta) \geq I(0, \pi/2) = \kappa(0, \pi/2) > 0.$$

□

Lemma 4.6.
$$\frac{\phi\kappa_\theta(\phi, t)}{\kappa(\phi, t)} < 1.$$

Proof. Our second integral formula is:

$$\cos(\theta)\kappa(\phi, t) - \sin(\theta)\kappa_\theta(\phi, t) = I(\phi, t).$$

The right hand side is positive, by the previous lemma. Hence

$$\cos(\theta)\kappa(\phi, t) > \sin(\theta)\kappa_\theta(\phi, t).$$

Hence
$$\frac{\phi\kappa_\theta(\phi, t)}{\kappa(\phi, t)} < \frac{\tan(\phi)\kappa_\theta(\phi, t)}{\kappa(\phi, t)} = \frac{\sin(\phi)\kappa_\theta(\phi, t)}{\cos(\phi)\kappa(\phi, t)} < 1.$$
This completes the proof. □

Now we are in a position to prove Lemma 4.4. Call this integral I_δ. Since $\sin(\theta) < \theta$ we have $I_\delta < J_\delta$, where

$$J_\delta = \int_0^\delta \frac{\phi}{\kappa(\phi, t)} d\phi. \tag{37}$$

Integrating by parts, we have

$$J_\delta = \frac{1}{2}\int_0^\delta \frac{\phi^2 \kappa_\theta(\phi, t)}{\kappa^2(\phi, t)} d\phi + \frac{\delta^2}{2\kappa(\delta, t)}. \tag{38}$$

Lemma 4.6 gives us

$$\frac{1}{2}\int_0^\delta \frac{\phi^2 \kappa_\theta(\phi, t)}{\kappa^2(\phi, t)} d\phi < \frac{1}{2}\int_0^\delta \frac{\phi}{\kappa(\phi, t)} d\phi = \frac{1}{2} J_\delta. \tag{39}$$

Combining this with Equation 38, we have

$$J_\delta < \frac{1}{2} J_\delta + \frac{\delta^2}{2\kappa(\delta, t)}. \tag{40}$$

Rearranging this equation gives us the result we seek.

5 The Fat Bowtie Theorem

In this chapter we prove the Fat Bowtie Theorem. Here it is again.

Theorem 5.1 (Fat Bowtie)**.** *Suppose that $C(0)$ is concinnous. Let $S \subset [-1, 1]^2$ denote the set of accumulation points of $L_t(C_t)$ as $t \to \infty$. If $(x, y) \in S$ and $|x| < 1$ then $|x| \le |y| \le 2|x| - x^2$.*

5.1 The Main Argument

Henceforth we assume that $C(t)$ is concinnous for all $t \in [0, T)$. Let us explain the geometric consequence of Equation 28. Let $C_+(t)$ be the positive lobe of $C(t)$. We write

$$C_+(t) = D(t) \cup V(t) \cup \overline{D}(t), \qquad \text{where} \tag{41}$$

- $D(t)$ connects $(0, 0)$ to $(x^*(t), Y(t))$.
- $V(t)$ connects $(x^*(t), Y(t))$ to $(x^*(t), -Y(t))$.
- $\overline{D}(t)$ connects $(x^*(t), -Y(t))$ to $(0, 0)$.

Figure 2 shows this decomposition.

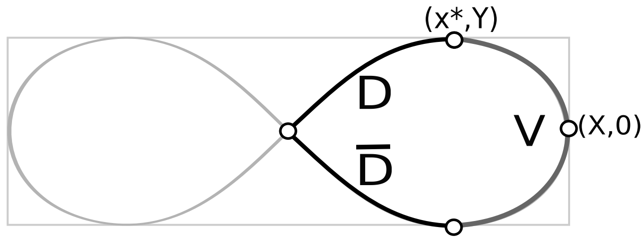

Figure 3: Decomposition of the right lobe.

The arc $L_t(V(t))$ is convex, and has endpoints which, by Equation 28, converge to $(1,1)$ and $(1,-1)$. Moreover $(1,0)$ is the midpoint of $L_t(V(t))$. From this structure, we see that $L_t(V(t))$ converges in the Hausdorff metric to the line segment connecting $(1,1)$ to $(-1,1)$. This is the right edge of the bowtie.

We say that a *monotone arc* is the graph G of a concave increasing function $g : [a,b] \to \mathbf{R}$ such that $g'(b) = 0$ and the unsigned curvature of G is strictly increasing going from left to right. (The derivatives are one-sided at the endpoints.) To G we associate the unique disk $E(G)$ such that $\partial E(G)$ contains the endpoints of G and the endpoint $(b, g(b))$ is the topmost point of $E(G)$. Figure 4 shows the construction.

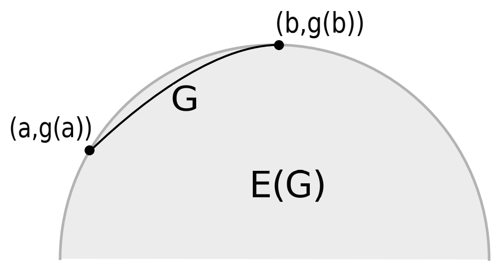

Figure 4: G and $E(G)$.

In the next section we prove the following result.

Lemma 5.2. *$G \subset E(G)$ whenever G is a monotone arc.*

The arc $D(t)$ is from the previous section is monotone. Hence $D(t) \subset E(t)$ where $E(t) = E(D(t))$. But then

$$L_t(D(t)) \subset L_t(E(t)). \tag{42}$$

The set $L_t(E(t))$ is a solid ellipse whose boundary contains the points $(0,0)$ and whose top point is $(x^*(t)/X(t), 1)$. The latter point converges to $(1,1)$ as $t \to T$. Moreover, by Lemma 3.2 the ratio of the vertical axis length of $L_t(E(t))$ to the horizontal axis length of $L_t(E(t))$ converges to ∞ as $t \to T$. The limit $\lim_{t\to T} L_t(E(t))$ must be the solid region under a downward pointing parabola whose boundary contains $(0,0)$ and whose top point is $(1,1)$. There is only one region with this description: It is given by the equation $y \leq 2x - x^2$.

This analysis shows that any accumulation point (x, y) of $L_t(D(t))$ has the form $x \leq y \leq 2x - x^2$. The Fat Bowtie Theorem now follows from symmetry.

We mention one other result that is proved using the same method.

Lemma 5.3 (Slope). *For any $\epsilon > 0$, the slope of $L_t(D(t))$ at $(0,0)$ lies in the interval $[1, 2+\epsilon]$ provided that t is sufficiently close to T.*

Proof. The lower bound on the slope is just convexity of the lobes of $C(t)$. Here is the proof of the upper bound, The slope of $\partial L_t(E(t))$ at $(0,0)$ converges to 2. Moreover, $L_t(D(t))$ starts at $(0,0)$ and lies beneath this boundary. Hence the slope of $L_t(D(t))$ at $(0,0)$ is at most $2 + \epsilon(t)$, where $\epsilon(t)$ is some function which tends to 0 as $t \to T$. □

To put the Slope Lemma in perspective, the Bowtie Conjecture would say that the slope of $L_t(D(t))$ converges to 1 as $t \to T$. So, at least the Slope Lemma picks out the same order of decay of the slope.

5.2 Proof of Lemma 5.2

We want to prove that $G \subset E(G)$ whenever G is a monotone arc. Let $E = E(G)$. We suppose this result is false and derive a contradiction. If this result is false then there is some arc δ of G whose endpoints p, q lie in ∂E but which is otherwise disjoint from E. The arc δ lies above E. We order the points so that p lies to the left of q. Let E^* be the osculating disk of G at q. Let r be the right endpoint of G. Figures 4,5,6 show these objects. There are three cases.

Case 1: Suppose $q = r$. Since G is monotone, G and E are tangent at q. Since G lies above q sufficiently close to q we see that the curvature of G at q is not more than the curvature of E. But then $E \subset E^*$. The Tait-Kneser Theorem tells us that $G - \{q\}$ is disjoint from E^*. This is impossible because $p \in E \subset E^*$.

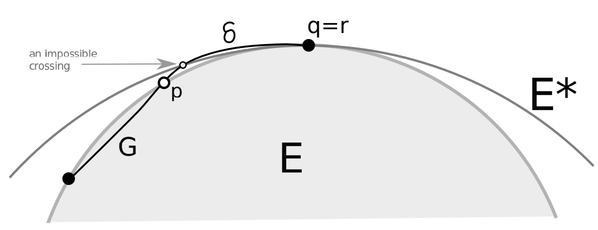

Figure 5: Case 1 of the argument.

Case 2: Suppose that $q \neq r$, and that G is tangent to E at q. There are two subcases. (A) If $E \subset E^*$ we get the same contradiction as in Case 1. (B) Otherwise E^* is a proper subset of E and the two disks are tangent at q. But then no point of E^* to the right of q intersects ∂E. But then the continuation of G to the right of q, which lies in E^* by the Tait-Kneser Theorem, cannot reach $r \in \partial E$.

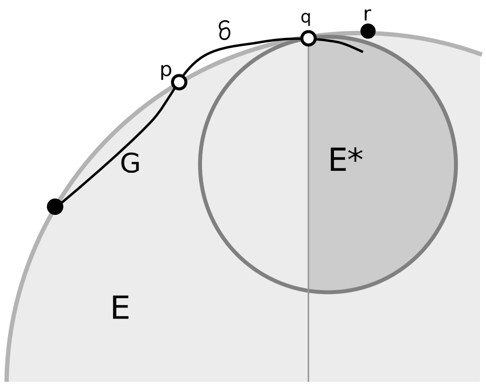

Figure 6: Case 2B of the argument.

Case 3: Suppose that $q \neq r$ and that G and ∂E are transverse at q. Since δ lies above E, we see that when we move rightward along ∂E^* we cross into the interior of E. Since δ lies above E, there must be some other point q', between p and q on E, where ∂E and G intersect. But two unequal circles can intersect at most twice. This means that all points of E^* to the right of q lie in the interior of E. Figure 6 shows the situation. This gives the same contradiction as in Case 2B.

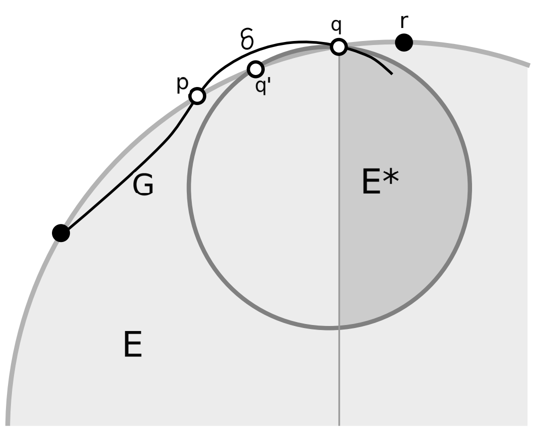

Figure 7: Case 3 of the argument.

This completes the proof

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