The Projective Heat Map

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Abstract

In this paper, we will introduce a projectively natural iteration on polygons which has a construction similar to the pentagram map but very different dynamical behavior. We will prove several results about the action of this map and also discuss some experimental observations on its interaction with the pentagram map.

1 Introduction

Perhaps the simplest polygon iteration works as follows. Starting with an n-gon P_1, we let P_2 be the n-gon whose vertices lie at the centers of the edges of P_1. One can iterate this construction, producing a sequence $\{P_n\}$ of polygons which shrink to a point. We can choose positive constants λ_n so that $\lambda_n P_n$ has unit diameter. For almost every choice of P_1, the rescaled sequence $\{\lambda_n P_n\}$ converges exponentially fast to an affinely regular n-gon. That is, the limit has the form $T(R_n)$, where T is an affine transformation and R_n is a regular polygon.

The proofs of the statements above are simple and beautiful. Define the primitive nth root of unity $\omega = \exp(2\pi i/n)$. One thinks of the vertices of the polygons as complex numbers. The map $M : P_1 \to P_2$ then becomes a complex linear map on $\boldsymbol{C}^n$. The kth eigenvector is the polygon W_k having vertices $\omega^k, ..., \omega^{nk}$. The two polygons W_1 and W_{n-1} are the eigenvectors corresponding to the largest eigenvector, and so $\lim \lambda_n P_n$ is a complex linear

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combination of W_1 and W_{n-1}; such combinations turn out to be affinely regular.

Switching back from $\boldsymbol{C}$ to $\boldsymbol{R}^2$, we note that the midpoint map M is affinely natural, in the sense that $M \circ T = T \circ M$ for any real affine map T of $\boldsymbol{R}^2$. The analysis above shows that the midpoint map is closely related to the discrete Fourier transform and to the heat equation. See [**Ta**] and [**Tr**] for connections between the midpoint map and (outer) billiards.

The purpose of this paper is to introduce a projectively natural variant of the M and to discuss some aspects of its behavior. Figure 1.1 shows the construction. One starts with an n-gon P_1 and produces a new n-gon $P_2 = H(P_1)$. The vertices of P_2 lie in the lines extending the edges of P_1, though not generally at the midpoints. We will give a more formal definition in the next chapter.

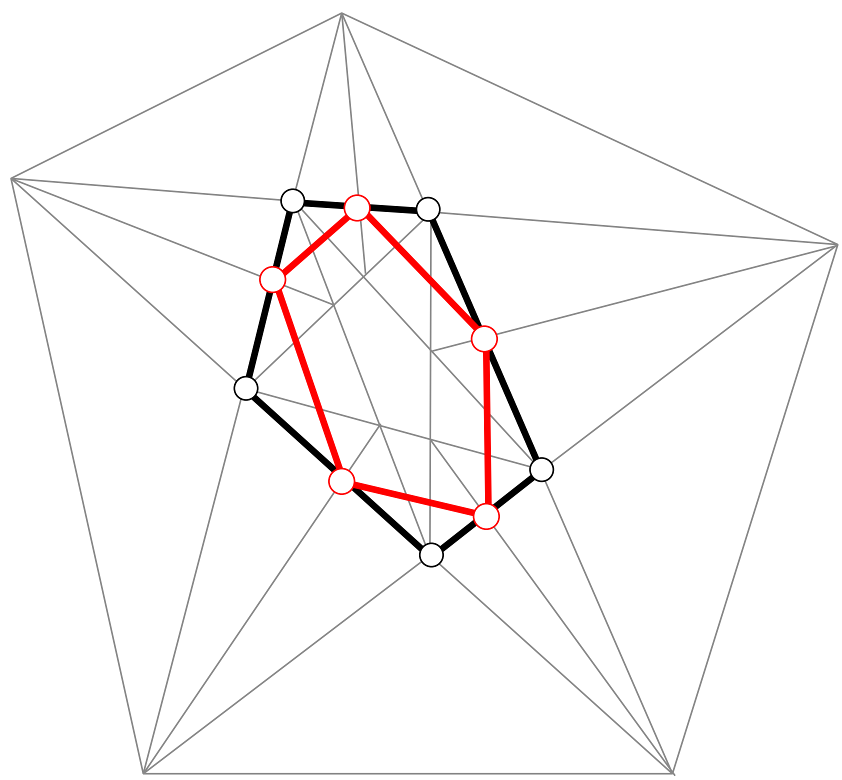

Figure 1.1: P_1 is black and $P_2 = H(P_1)$ is red.

The map H is defined entirely in terms of projective geometry constructions – i.e., taking the line between two points and taking the intersection of two lines. For this reason, H is projectively natural. That is, $H \circ T = T \circ H$ for any projective transformation T. The map H makes sense over any field, though there needs to be enough points in the field for the construction to make sense. We shall mainly be concerned about the action of H on real polygons. Figure 1.1 shows the action of H on a convex pentagon. For non-convex real polygons, H is generically well defined.

We call H the *projective heat map*. The projective heat map has more symmetry than the midpoint map but it is nonlinear. For this reason, it has a much more interesting and complicated structure.

The map H might remind some readers of the pentagram map, and indeed I thought of the map H in response to the recent flurry of work on the pentagram map. The case $n = 5$ of the pentagram map is classical; it goes back at least to Clebsch in the 19th century and perhaps even to Gauss. Motzkin [**Mot**] also considered this case in 1945. I studied the pentagram map on general polygons in [**Sch1**], [**Sch2**], and [**Sch3**]. In recent years, the pentagram map has been a topic in a variety of papers, thanks to its (now known) complete integrability and its connection to cluster algebras. See for instance [**OST1**], [**OST2**], [**Sol1**], [**KS**], [**Gli**], [**GSTV**], [**MB1**], [**MB2**].

While the action of H is very different from the action of the pentagram map, one can see from Figure 1.1 that the construction of H is related to the pentagram map. Each point of $H(P_1)$ is the intersection of a line extending an edge of P_1 with a line through a vertex of $\Pi(P_1)$ and the corresponding vertex of $\Pi^{-1}(P_1)$. Here Π and Π^{-1} are the pentagram map and its inverse.

The map H is also reminiscent of the map I studied in [**Sch4**]. The map in [**Sch4**] is a projectively natural iteration defined on convex polygons inscribed in the circle. It seemed hard to define this map for non-convex polygons, and it makes no sense at all for polygons which are not inscribed. So, one could view H as a robust variant of the map in [**Sch4**].

In this paper, I will prove several easy results about H and also describe some experimental observations concerning the interaction between H and the pentagram map. Here is the main result of this paper.

Theorem 1.1 *Let $\mathcal{C}_5$ denote the space of projective equivalence classes of convex pentagons. For any $x \in \mathcal{C}_5$, the sequence $\{H^n(x)\}$ converges to the projectively regular class.*

The proof I give involves finding an increased quantity for H on $\mathcal{C}_5$. The increased quantity turns out to be the main Casimir for the Poisson bracket associated to the pentagram map. Since this proof makes the result look like an algebraic accident, I will sketch a second, more robust proof, based on complex analysis.

Let $\mathcal{P}_5$ denote the space of projective equivalence classes of (not necessarily convex) real pentagons. We define the *Julia set* $\mathcal{J}_5$ to be the set of projective classes of pentagons such that $H^n(x)$ is non-convex for all n. In Figure 1.2, we color points of $\mathcal{P}_5$ (which we identify with the plane) according to how many iterates it takes for them to land in $\mathcal{C}_5$. The longer it takes, the darker the point. set $\mathcal{J}_5$ is the "infinitely dark" region. In a forthcoming paper, I will show that $\mathcal{J}_5$ is a connected set of measure 0.

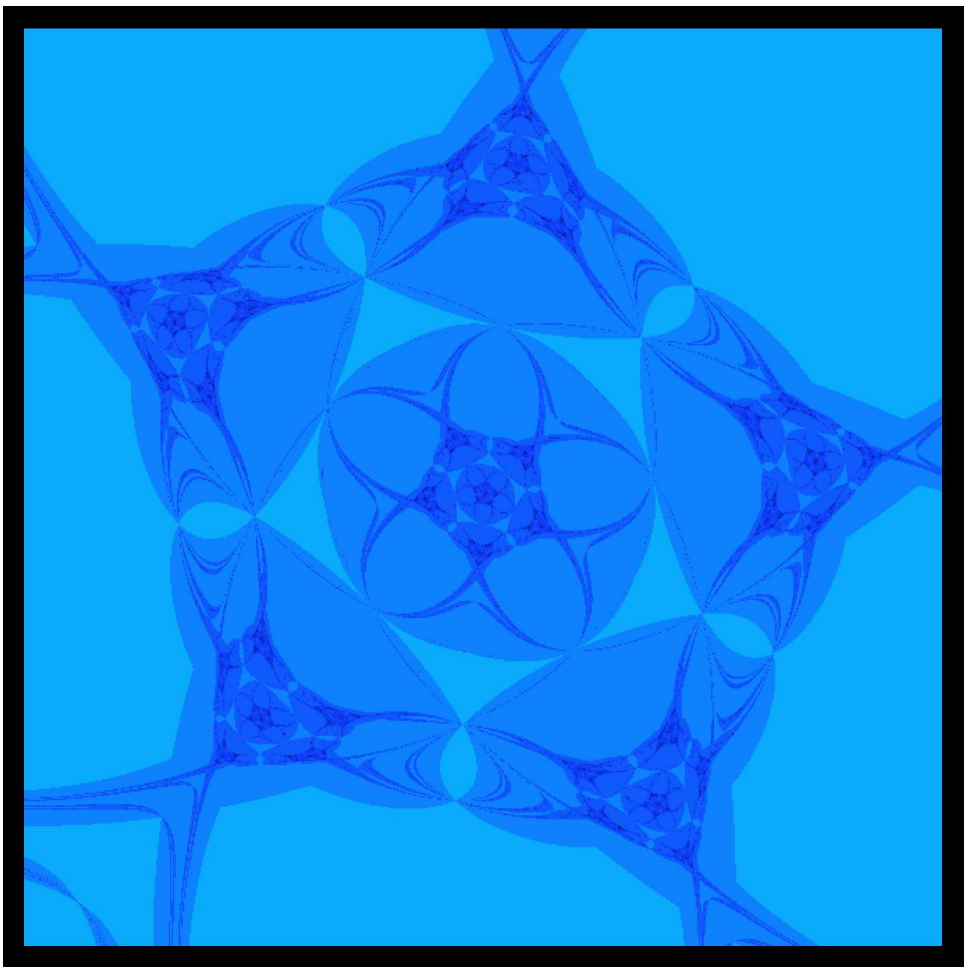

Figure 1.2: Part of the Julia set $\mathcal{J}_5$.

Theorem 1.1 is part of a bigger conjecture. Let $\mathcal{C}_n$ and $\mathcal{P}_n$ be the obvious generalizations of $\mathcal{C}_5$ and $\mathcal{P}_5$ to n dimensions.

Conjecture 1.2 *The following is true for all $n \geq 5$.*

1. *For any $x \in \mathcal{C}_n$, the sequence $\{H^n(x)\}$ converges to the projectively regular class.*
2. *For almost every $x \in \mathcal{P}_n$, the sequence $\{H^n(x)\}$ converges to the projectively regular class.*

Theorem 1.1 takes care of Statement 1 in the case $n = 5$. The measure 0 result mentioned above proves Statement 2 in the case $n = 5$. In this paper I will give one reason why Statement 1 of Conjecture 1.2 is harder for $n > 5$ than it is for $n = 5$. At the end of §2 we will prove the following pessimistic result.

Theorem 1.3 *For $n \geq 6$ even, the map H has no increasing self-dual quantities.*

As we will explain in §3.2, projective duality gives (up to cyclic relabeling) a natural involution on the spaces $\mathcal{C}_n$ and $\mathcal{P}_n$. A *self-dual quantity* is a function $f : \mathcal{C}_n \to \mathcal{C}_n$ which is invariant under both cyclic relabeling and the duality involution.

In §4, I will compare the projective heat map to several other polygon iterations, and discuss the interaction between the pentagram map and the projective heat map. The idea is to consider the semigroup generated by the projective heat map, the pentagram map, and projective dualities. By considering certain words in this semigroup, I will produce maps which seem to exhibit a mixture of integrability and heat-like (parabolic) behavior. There are no theorems in §4, just observations and conjectures.

I started thinking about the pentagram map during the excellent conference FDIS '13 held in Luminy in July 2013. There was quite a bit of talk about the pentagram map, and I decided that people at the conference might like the projective heat map as well even though it is not really an integrable system. I would like to thank Stergios Antonakoudis, Max Glick, Nhat Le Quang, Gloria Mari-Beffa, Curt McMullen, Sergei Tabachnikov, and Guilio Tiozzo for helpful comments and suggestions.

2 Preliminaries

2.1 Projective Geometry

Most of what we say works over any field, though we stick to the reals for ease of exposition. The *real projective plane* $\boldsymbol{RP}^2$ is the space of lines through the origin in $\boldsymbol{R}^3$. Such lines are denoted by $[x : y : z]$. This is the line consisting of all vectors of the form (rx, ry, rz) with $r \in \boldsymbol{R}$.

In the usual way, we think of $\boldsymbol{R}^2$ as an affine patch of $\boldsymbol{RP}^2$. Concretely, the inclusion is given by

$$(x, y) \to [x : y : 1]. \tag{1}$$

A *line* in $\boldsymbol{RP}^2$ is a collection of points represented by all the lines in a plane through the origin in $\boldsymbol{R}^3$. The set $\boldsymbol{RP}^2 - \boldsymbol{R}^2$ is a single line, called the *line at infinity*. All other lines in $\boldsymbol{RP}^2$ intersect $\boldsymbol{R}^2$ in a straight line. Conversely, any straight line construction in $\boldsymbol{R}^2$ extends naturally to a straight line construction in $\boldsymbol{RP}^2$. When we make our constructions, we will draw things in the plane (of course) but we really mean to make the constructions in the projective plane.

A *projective transformation* is a self-homeomorphism of $\boldsymbol{RP}^2$ induced by the action of an invertible linear transformation. Projective transformations permute the lines of $\boldsymbol{RP}^2$ and are in fact analytic diffeomorphisms. Conversely any homeomorphism of $\boldsymbol{RP}^2$ which carries lines to lines is a projective transformation.

A subset of $\boldsymbol{RP}^2$ is *convex* if some image of that subset under a projective transformation is a convex subset of $\boldsymbol{R}^2$ in the ordinary sense. For instance, a hyperbola in the plane extends to a closed loop in $\boldsymbol{RP}^2$ which bounds a convex subset on one side.

The space $(\boldsymbol{RP}^2)^*$ of lines in $\boldsymbol{RP}^2$ is known as the *dual space*. A *projective duality* is a map from $\boldsymbol{RP}^2$ to $(\boldsymbol{RP}^2)^*$ which maps collinear points to coincident lines. One example of a duality is as follows. Starting with a point $P \in \boldsymbol{RP}^2$ we let L_P be the line through the origin in $\boldsymbol{R}^3$ representing P. We map P to the line in $\boldsymbol{RP}^2$ represented by the orthogonal plane $(L_P)^\perp$. Any other duality is a composition of this one with a projective transformation.

Pentagons are self-dual. Suppose that P is a pentagon having vertices P_1, P_2, P_3, P_4, P_5. We define $L_k = \overline{P_{k+2}P_{k-2}}$, with indices taken mod 5. Then there is a (single) projective duality which simultaneously carries P_k to L_k for $k = 1, 2, 3, 4, 5$.

2.2 Flag Invariants

Now we will introduce coordinates on the space $\mathcal{P}_n$. These coordinates are used in [**S1**], [**S2**], [**S3**], [**OST1**], and [**OST2**] for the pentagram map. In these papers, they are called *corner invariants*. However, as in [**S5**], it seems better to call them *flag invariants*.

The *inverse cross ratio* of 4 real numbers $a, b, c, d \in \boldsymbol{R}$ is the quantity

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-c)(b-d)} \tag{2}$$

When $a < b < c < d$, the quantity $[a, b, c, d]$ lies in $(0, 1)$. We will usually consider this situation.

Given 4 collinear points A, B, C, D in the projective plane, we choose some projective transformation which identifies these points with 4 numbers on the x-axis, and then we take the cross ratio of the first coordinates of these numbers. This lets us define $[A, B, C, D]$. The result is independent of any choices made. Moreover, the cross ratio is invariant under projective transformations.

On an oriented polygon, a *flag* is a pair (v, e) where v is a vertex of the path and e is an edge of the path. We indicate the flag (v, e) with an auxilliary point placed on the edge e two-thirds of the way towards v. Figure 2.1 shows what we mean.

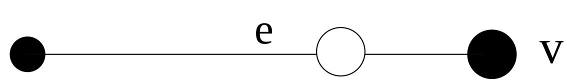

Figure 2.1: Denoting the flag (v, e).

Suppose we have an oriented polygon, as shown in Figure 2.2. We orient the flags according to the following scheme.

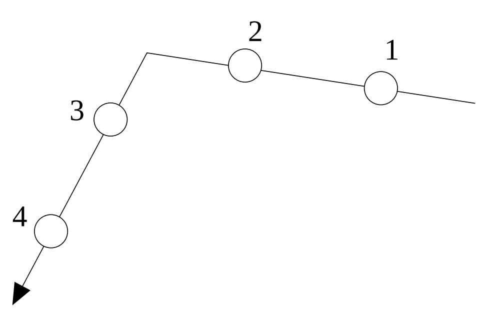

Figure 2.2: Ordering the flags along an oriented path

Finally, to each flag along the polygon, we associate the cross ratio of the associated points shown in Figure 2.3. This picture is meant to be invariant under projective transformations. We call these the *flag invariants*.

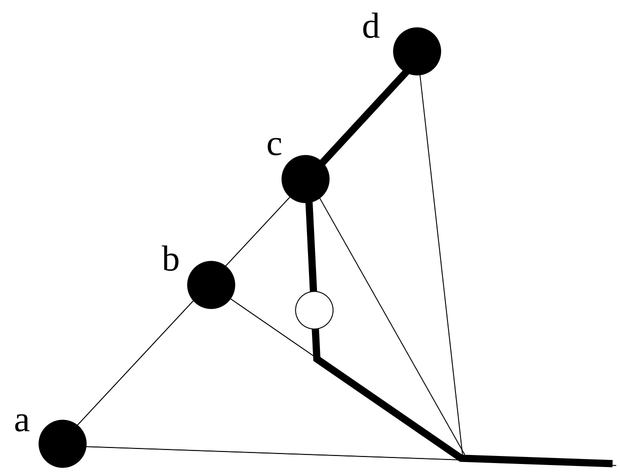

Figure 2.3: Invariant of a flag

Let us consider the naturality of this construction. The cross ratio of interest can be computed in two ways. First of all, it is the cross ratio of the 4 points shown. Two of the points involved are adjacent to the flag point. On the other hand, the cross ratio can be computed as the cross ratio of the 4 drawn lines. Two of the lines are adjacent to the line of the flag, going in the other direction from the abovementioned points. The picture is invariant not just under projective transformations but also projective dualities.

Pentagons: For pentagons, which are self-dual, the invariants have the form $(x_1, ..., x_5, x_1, ..., x_5)$. This comes from the self-dual property of pentagons. Any two consecutive invariants determine the rest, a direct calculation reveals that

$$(x_{k+1}, x_{k+2}) = G(x_k, x_{k+1}), \qquad G(x, y) = \Big(y, \frac{1-x}{1-xy}\Big). \tag{3}$$

The order 5 birational map G is sometimes called the *Gauss recurrence*. For $n = 5$, we prefer to work with the square root of the main Casimiar $x_1x_2x_3x_4x_5$. This is the increased quantity mentioned in connection with Theorem 1.1.

2.3 The Projective Energy

Given $P \in \mathcal{C}_n$, we define

$$E(P) = \prod_{i=1}^{2n} x_i, \tag{4}$$

where $x_1, ..., x_{2n}$ are the flat invarants of P. In the language of [**Sch3**, [**OST1**], and [**OST2**], we have

$$E(P) = O_n(P)E_n(P), \tag{5}$$

where O_n and E_n are the Casimirs for the pentagram invariant Poisson structure. We call E the *projective energy*. When $n = 5$ we prefer to redefine $E(P) = x_1x_2x_3x_4x_5$, because of the redundancy in the coordinates mentioned at the end of the last section.

In this section we recall a geometric interpretation of the projective energy. Any convex domain $X \subset \boldsymbol{RP}^2$ comes equipped with a canonical metric, known as the *Hilbert metric*. Figure 2.4 illustrates the construction. The distance between the points b and c is given by

$$d(b, c) = -\log[a, b, c, d]. \tag{6}$$

Here $[a, b, c, d]$ is the inverse cross ratio, as above.

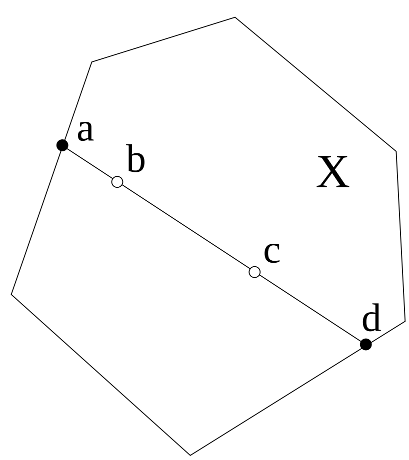

Figure 2.4: Definition of the Hilbert Metric

Here a and b are there the line $\overline{b}c$ intersects ∂X. It is well known, and a fairly easy exercise, to show that this formula really does satisfy the triangle inequality.

If $Y \subset X$ is a polygon, we define $\Pi_X(Y)$ to be the perimeter of Y with respect to the Hilbert metric on X. We have

$$E(P) = \exp(\Pi_P(Q)), \tag{7}$$

where Q is the image of P under the pentagram map, as shown in Figure 2.5.

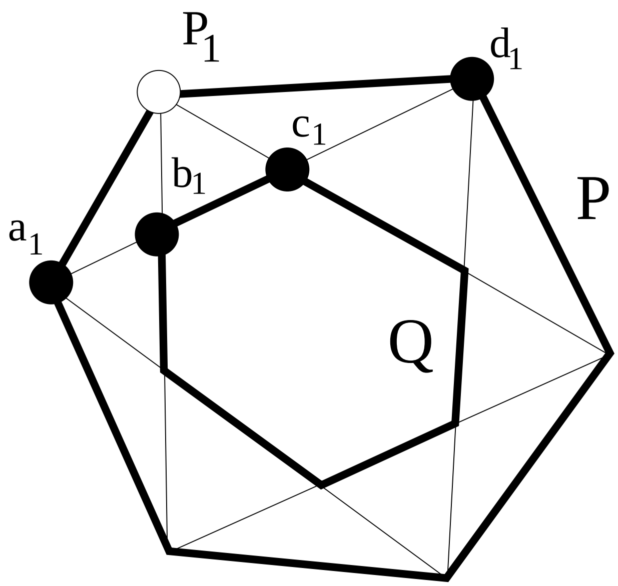

Figure 2.5: The pentagram map

The derivation of Equation 7 is straightforward. Referring to Figure 2.5, we have

$$\exp(\Pi_P(Q)) = \prod_{i=1}^{n} \xi_i, \qquad \xi_i = [a_i, b_i, c_i, d_i]. \tag{8}$$

Here ξ_i is associated to the vertex P_i of P. This is shown For $i = 1$ in Figure 2.5. A direct calculation shows that $\xi_j = x_{2j-1}x_{2j}$, where x_{2j-1} and x_{2j} are the two flag invariants associated to the flags involving P_j.

3 General Formula for the Heat Map

3.1 Definition of the Heat Map

In general, we start with an n-gon P, having vertices $P_0, P_2, P_4, \ldots$ and produce a new n-gon $Q = H(P)$ having vertices $Q_1, Q_3, Q_5, \ldots$. It is useful to label the vertices of P and Q with indices of the opposite parity. (The next iterate $R = H(Q)$ would again be labeled by even integers. See the remark at the end of this section.) Figure 2.4 shows how we can choose a point Q_3 in a projectively natural way from the edge P_2P_4 of P using the 4 points P_0, P_2, P_4, P_6 as guides.

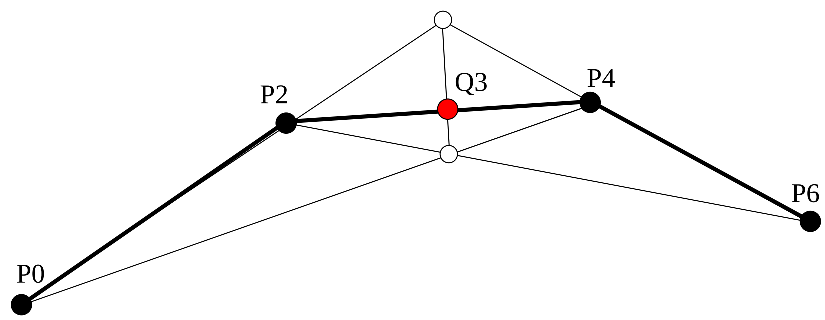

Figure 2.4: Choosing a point on an edge.

The point Q_3 has an alternate description. There is some projective transformation T such that, relative to the usual affine patch,

- $T(P_0) = (1, -1)$.
- $T(P_2) = (1, 1)$.
- $T(Q_3) = (0, 1)$.
- $T(P_4) = (-1, 1)$.
- $T(P_6) = (-1, -1)$.

We produce the rest of the points of Q by performing the same construction on the points with shifted indices. Our construction makes sense over essentially any field, is projectively natural, and respects convexity in the real case.

Considered as a map on the space of labeled polygons, really only the map H^2 has a canonical definition. However, as we explain in the next chapter, for pentagons, there is a canonical labeling scheme for H itself.

3.2 The Dual Heat Map

We choose some projective duality Δ. Up to a choice of labeling scheme, Δ defines a diffeomorphism $\Delta : \mathcal{P}_n \to \mathcal{P}_n$ as follows. Δ maps the projective class of the n-gon P to the projective class of the n-gon Q with the following property: In the correct cyclic order, the lines extending the edges of Q are the images of the vertices of P under Δ. The action on $\mathcal{P}_n$ does not depend on the choice of duality.

We define the *dual heat map* H^* by the formula

$$H^*(P) = \Delta \circ H \circ \Delta(P). \tag{9}$$

Were we to make our construction for the pentagram map in place of the heat map, we would produce the inverse pentagram map. This leads one to surmise that H and H^* are inverses of each other. However, this is not the case.

3.3 The Reconstruction Formula

In general, if one wants to derive the equation for some projectively natural iteration, in terms of the flag invariants, one could undertake the following 3-step process.

1. Start with the flag invariants and reconstruct the polygon.
2. Apply the desired map and get a new polygon.
3. Compute the flag invariants of the new polygon.

The third step is explained in the previous chapter. In this section we will explain the first step. We quote a result from [**Sch3**]. The proof we give in [**Sch3**] is rather involved, but for our purposes we only need the first 9 points of our polygon to derive a formula for the projective heat map. In this case, the reader can verify by direct calculation that enough of our formula is correct for the purposes here.

Our polygon will have vertices P_{9+2k} for $k = -8, -6, -4, -2, \ldots$. We normalize so that (in homogeneous coordinates)

$$P_{-7} = \begin{pmatrix} 0 \\ x_0 x_1 \\ 1 \end{pmatrix}, \quad P_{-3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{10}$$

We define polynomials O_a^b for $a \leq b+2$ odd integers, in the following recursive way. First, we define $O_{-1}^{-1} = 1$ and $O_{b+2}^b = 0$. Next, we define

$$O_b^b = O_{b-2}^b = 1, \qquad O_a^b = \begin{pmatrix} 1 \\ -x_{b-2} \\ x_{b-4}x_{b-3}x_{b-2} \end{pmatrix} \cdot \begin{pmatrix} O_a^{b-2} \\ O_a^{b-4} \\ O_a^{b-6} \end{pmatrix}, \qquad a = b-4, b-6, ... \tag{11}$$

The rest of the points of the polygon are given by

$$P_{9+2k} = \begin{pmatrix} O_{-1}^{3+k} \\ O_{+1}^{3+k} \\ O_{+3}^{3+k} \end{pmatrix}, \qquad k = 0, 2, 4, ... \tag{12}$$

The list of flag invariants starts out $x_0, x_1, x_2, ...$

3.4 The Formula for the Heat Map

The most important thing we want to point out is that there is not a canonical formula for H. One must break symmetry and shift the indices one unit (or more) to the left or to the right. There is, however, a canonical formula for H^2. Readers who are familiar with the pentagram map will recognize and understand the situation immediately, because the same thing happens there.

What we will do is compute a formula for a map which we call Ω. It turns out that $\Omega^2 = H^*H$. We also have the left and right shift maps L and R, which respectively shift the indices one unit to the left or to the right. These maps act on $\boldsymbol{R}^{2n}$ and have order $2n$. Here are some relations amongst these maps.

- Both $L\Omega$ or $R\Omega$ give formulas for H with some labeling convention.
- Both ΩL or ΩR give formulas for H^* with some labeling convention.
- The map $L\Omega R\Omega = R\Omega L\Omega$ gives a canonical formula for H^2.
- The map $\Omega R\Omega L = \Omega L\Omega R$ gives a canonical formula for $(H^*)^2$.
- Up to suitable labeling, we have $H = H^*$ on $\mathcal{P}_5$. In this case Ω itself gives a canonical formula for H.

To compute the formula for Ω, we write out the first 8 points of P, construct the first 5 points of Q using the straight line construction, and then compute the only 2 flag invariants of Q we have enough information to compute. We did the calculation in Mathematica, and then tried to make the answer as symmetric as we could.

Suppose that P has projective flag invariants $x_1, x_2, ...$, as above. Suppose that $Q = H(P)$ has flag invariants $y_1, y_2, y_3, ...$ Define

$$A_k = 2 + x_{k-3/2} + x_{k+3/2}. \tag{13}$$

$$B_k^{\pm} = A_{k\mp 1} - x_{k\pm 1/2}A_{k\pm 1} + x_{k\pm 5/2}A_{k\mp 1}. \tag{14}$$

$\Omega(x_1, ..., x_{2n}) = (y_1, ..., y_{2n})$, where

$$\frac{y_{2k+0}}{x_{2k+1}} = \frac{A_{2k-5/2}B^{-}_{2k+1/2}}{B^{+}_{2k-3/2}B^{+}_{2k+1/2}}, \qquad \frac{y_{2k+1}}{x_{2k+0}} = \frac{A_{2k+5/2}B^{+}_{2k+1/2}}{B^{-}_{2k+3/2}B^{-}_{2k+1/2}}. \tag{15}$$

Of course, one can solve for y_{2k+0} and y_{2k+1}.

It seems worth unpacking this formula a bit. Concretely, we have

$$y_5 = \frac{x_6(2 + x_1 + x_4)(-2 - 2x_3 + x_5 + x_5x_6 - x_8 - x_3x_8)}{(2 + x_1 - x_4 - x_3x_4 + 2x_6 + x_1x_6)(-2 - x_3 + x_6 + x_5x_6 - 2x_8 - x_3x_8)}.$$

The formula for y_6 is obtained from the formula for y_5 by replacing each index k by $11 - k$. The formula for the remaining y variables is obtained by cyclically shifting the formulas for y_5 and y_6 by suitably chosen even amounts.

Proof of Theorem 1.3: We argue by contradiction. Let g be a self-dual increased quantity. Since g is an increased quantity for H, then g is also an increased quantity for the dual map H^*. This would mean that H^*H has no other fixed points in $\mathcal{C}_n$ besides the projectively regular class.

We compute easily that $\Omega(a, a, b, b, a, a, b, b, ...) = (b, b, a, a, b, b, a, a, ...)$, when the length of the string is divisible by 4. Hence $\Omega^2 = H^*H$ fixes all such points. These points correspond to $2k$-gons with $(2k)$-fold dijedral symmetry. This shows that H^*H has many other fixed points in $\mathcal{C}_n$ when $n \geq 6$ is even. ♠

Remark: Computer experiments for $n \geq 7$ odd suggest that H^*H also has non-regular fixed points in $\mathcal{C}_n$ in these cases as well. Thus, Theorem 1.3 should work for all $n \geq 6$.

4 The Action on Pentagons

4.1 The Space of Convex Pentagons

We identify $\mathcal{P}_5$ with $\boldsymbol{R}^2$ by the map

$$\Psi(P) = (x, y), \qquad x = x_1(P), \qquad y = x_2(P). \tag{16}$$

Here $x_1(P)$ and $x_2(P)$ are the first two flag invariants of the pentagon class P. We sometimes write $P(x, y)$ for the polygon with coordinates $x = x_1(P)$ and $y = x_2(P)$.

Let D_5 be the order 10 group generated by the Gauss recurrence and by the reflection $(x, y) \to (y, x)$. Our map Ψ is closely related to the action of D_5. The pentagons P and P' are pentagons related by some dijehedral relabelling if and only if $\Psi(P)$ and $\Psi(P'$ are in the same D_5-orbit.

The flag invariants for a convex polygon all lie in $(0, 1)$, so $\Psi(\mathcal{C}_5) \subset (0, 1)^2$.

Lemma 4.1 *Ψ is a homeomorphism from $\mathcal{C}_5$ onto $(0, 1)^2$.*

Proof: The flag invariants for a convex polygon all lie in $(0, 1)$. Therefore $\Psi(\mathcal{C}_5) \subset (0, 1)^2$. Since the first two invariants determine the pentagon, the map Ψ is injective. Also, Ψ is continuous because all the points and lines in the construction of the invariants vary continuously with the polygon.

Now we show that Ψ is surjective. Given any $(x, y) \in (0, 1)^2$ we can produce a list $(x_1, ..., x_{10})$ of invariants and then reconstruct the polygon $P(x, y)$. We consider a path in $(0, 1)^2$ which starts at R_0 and ends at our point (x, y). As we move along this path, the created pentagons stay in general position throughout. (If not, then some flag invariant would lie in $\{0, 1, \infty\}$.) Since the pentagons are convex at one end of the homotopy, they are convex at the other end. Hence, the pentagon $P(x, y)$ is convex. This proves that Ψ is surjective.

Our construction of $P(x, y)$ from x and y depends continuously on x and y. Hence Ψ^{-1} is continuous. ♠

Remark: When $n > 5$ we get an embedding of $\mathcal{P}_n$ into $(0, 1)^{2n-8}$. In general, this embedding is not onto. For instance, when $n = 10$, the cube $(0, 1)^{12}$ also contains the flag invariants corresponding to locally convex 10-gons which wind twice around a convex pentagon.

4.2 Formula in the Pentagon Case

We first describe a canonical labeling scheme for the map H, acting on labeled pentagons. This time we label the pentagon vertices by integers $1, 2, 3, 4, 5$. If $Q = H(P)$ the ith vertex of Q lies in the edge of P that is diametrically opposite to P, in a combinatorial sense. This is the most natural possible labeling convention. We adopt this labeling convention for the rest of this chapter.

The identification of $\mathcal{P}_5$ with $\boldsymbol{R}^2$ converts the projective heat map into a rational map $H : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. We derive a formula for H by a three step process.

- We express the flag invariants $x_2, x_3, ...$ in terms of $x = x_1$ and $y = x_2$ using the Gauss recurrence.
- We apply the general formula for the projective heat map given in Equation 15.
- We take the first two coordinates of the result; this is $H(x, y)$.

We do the calculation in Mathematica, and it gives us the following result.

$$\begin{aligned} H(x,y) &= (x', y'), \\ x' &= \frac{(xy^2 + 2xy - 3)(x^2y^2 - 6xy - x + 6)}{(xy^2 + 4xy + x - y - 5)(x^2y^2 - 6xy - y + 6)} \\ y' &= \frac{(x^2y + 2xy - 3)(x^2y^2 - 6xy - y + 6)}{(x^2y + 4xy - x + y - 5)(x^2y^2 - 6xy - x + 6)} \end{aligned} \tag{17}$$

4.3 Fixed Points

Let $\phi = (1+\sqrt{5})/2$ denote the golden ratio. It turns out that H has 1 attracting fixed point $(1/\phi, 1/\phi)$, corresponding to the regular class, and 6 repelling fixed points. One of the repelling fixed points is $(-\phi, -\phi)$, corresponding to the star regular class. The 5 additional fixed points are

$$(-8, -1/2), (-1/2, -3), (-3, -3), (-3, -1/2), (-1/2, -8). \tag{18}$$

These points lie in a single orbit of G, as they must. They correspond to the projective class of the non-convex isosceles pentagon with coordinates

$$(0, 0), \quad (1, 0), \quad (1, 1), \quad (1, 0), \quad (2/3, 1/2).$$

We compute

$$dH(\frac{1}{\phi},\frac{1}{\phi}) = \begin{bmatrix} \frac{2}{\phi^5} & 0 \\ 0 & \frac{2}{\phi^5} \end{bmatrix}, \qquad dH(-\phi,-\phi) = \begin{bmatrix} \frac{-\phi^5}{2} & 0 \\ 0 & \frac{-\phi^5}{2} \end{bmatrix}. \tag{19}$$

Beautifully, the product of the two matrices is $-I$. The fact that the second matrix is a similarity is probably what explains the small-scale self-similar structure $\mathcal{J}_5$ shown in Figure 1.2.

Finally, here is the linearization about one of the other fixed points.

$$dH(-3,-3) = \begin{bmatrix} -2/3 & 20/3 \\ 20/3 & -2/3 \end{bmatrix} \tag{20}$$

This matrix has eigenvalues $-22/3$ and 6. So, about the point $(-3,-3)$, the map H repels different directions at different rates. The eigenvalues of dH at the remaining fixed points are the same as at $(-3,-3)$, thanks to the symmetry coming from the Gauss recurrence.

4.4 Positive Dominant Polynomials

Here we prove a result about polynomials which will be useful in the next section. This result is probably well known, but I formulated and proved it myself in [**Sch4**]. I don't know of another reference.

We consider real polynomials in the variables $x_1,...,x_k$. Given a multi-index $I=(i_1,...,i_k)\in(\boldsymbol{N}\cup\{0\})^k$ we let

$$x^I = x_1^{i_1}...x_k^{i_k}. \tag{21}$$

Any polynomial $F\in\boldsymbol{R}[x_1,...,x_k]$ can be written succinctly as

$$F=\sum A_I X^I, \qquad A_I\in\boldsymbol{R}. \tag{22}$$

If $I'=(i_1',...,i_k')$ we write $I'\le I$ if $i_j'\le i_j$ for all $j=1,...,k$. We call F *weak positive dominant* if

$$\sum_{I'\le I} A_{I'}\ge 0 \qquad \forall I, \tag{23}$$

We call F *positive dominant* if the total sum, $\sum a_{ij}$, is positive. When $k=0$, *positive dominant* means *a positive number*.

Lemma 4.2 *If F is weak positive dominant then $F \geq 0$ on $(0,1)^k$. If F is positive dominant, then $F > 0$ on $(0,1)^k$.*

Proof: Write

$$F = f_0 + f_1 x_k + ... + f_m x_k^m, \qquad f_j \in \boldsymbol{R}[x_1, ..., x_{k-1}]. \tag{24}$$

Let $F_j = f_0 + ... + f_j$. The positive dominance of F implies the weak positive dominance of F_j for all j and the positive dominance of F_m. By induction, $F_j \geq 0$ on $(0,1)^{k-1}$. Let $x_k \in (0,1)$. Since $x_k^i \geq x_k^j$ for $i < j$,

$$F = f_0 + f_1 x_k + ... + f_m x_k^m \geq F_1 x_k + f_2 x_k^2 + ... + f_m x_k^m \geq$$
$$F_2 x_k^2 + f_3 x_k^3 + ... + f_m x_k^m \geq \ldots \geq F_m x_k^m > 0. \tag{25}$$

The last term is positive by induction. ♠

4.5 Dihedral Symmetry

Let D_5 denote the order 10 dihedral group generated by the Gauss recurrence and by the coordinate swap $(x,y) \to (y,x)$. The action of D_5 on $\mathcal{P}_5$ is induced by the operation of dihedrally relabelling the pentagons.

The Julia set $\mathcal{J}_5$ contains the point $(1,1)$, and for this reason we want to stay away from this point. Here is a technical result we will use later.

Lemma 4.3 *The open square $(0,1/\phi)^2$ contains a fundamental domain for the action of D_5 on $(0,1)^2 - (1/\phi, 1/\phi)$.*

Proof: We compute

$$G^2(x,x) = (\frac{1}{1+x}, 1 - x^2). \tag{26}$$

We easily compute that G^2 maps the segment connecting $(1/\phi, 1/\phi)$ to $(1,1)$ to a curve γ which connects $(1/2, 0)$ to R_0 and remains inside $(0,1/\phi)^2$ except at the endpoint $(1/\phi, 1/\phi)$. But then a fundamental domain for the action of D_5 on $(0,1)^2 - (1/\phi, 1/\phi)$ is contained in the open region bounded by the segment connecting $(0,0)$ to $(1/\phi, 1/\phi)$, the segment connecting $(0,0)$ to $(1/2, 0)$, and γ. This region is contained in $(0,1/\phi)^2$. ♠

4.6 Existence of an Increased Quantity

Given some $P = P(x,y)$, we write the projective energy of P like this.

$$E(P) = x_1x_2x_3x_4x_5 = \frac{xy(x-1)(y-1)}{1-xy}. \tag{27}$$

Theorem 4.4 *$E(H(P)) \geq E(P)$ for all $P \in \mathcal{C}_5$, with equality if and only if P is the projectively regular class.*

Before we prove Theorem 4.4, we use the result to prove Theorem 1.1.

Proof of Theorem 1.1: Let $R = (1/\phi, 1/\phi)$ be the projectively regular class. Theorem 1.1 is equivalent to the following statement. Given any $p \in (0,1)^2$, the sequence $H^n(p)$ converges to R. Let $p_n = H^n(p)$ and let $E_n = E(p_n)$. Note that $\{E_n\}$ is non-decreasing. It follows directly from Equation 27 that that the level sets $E^{-1}[E_1, 1]$ are compact. Hence $\{p_n\}$ stays within a compact subset of $(0,1)^2$. On a subsequence, we have $p_n \to q$. By construction $E_n \to E(q)$. But if $q \neq R$ then $E_n \to E(H(q)) > E(q)$, and this is a contradiction. Hence $q = R$. ♠

Now we prove Theorem 4.4. Here $x = x_1$ and $y = x_2$, and x_3, x_4, x_5 are the next three flag invariants of P. Using Mathematica [**W**], we compute that

$$E(H(P)) = \frac{(-1+xy)(-3-x+xy)(-3-y+xy)}{(4+x+y)(-5-x+y+4xy+x^2y)(-5+x-y+4xy+xy^2)}$$

$$\times \frac{(-4+x+y+2xy)(-3+2xy+x^2y)(-3+2xy+xy^2)}{(6-x-6xy+x^2y^2)(6-y-6xy+x^2y^2)} \tag{28}$$

We have split things up this way simply because the equation is too long to fit on one line. Again using Mathematica, we compute

$$E(H(P)) - E(P) = \frac{N(x,y)}{D(x,y)} \tag{29}$$

Where $D(x,y)$ is some polynomial whose composition is not important to us and $N(x,y)$ is the following polynomial.

$$
\begin{gathered}
-x^{10}y^9 - 3x^{10}y^8 + 3x^{10}y^7 + x^{10}y^6 \\
-x^9y^{10} - 10x^9y^9 - 4x^9y^8 + 66x^9y^7 - 34x^9y^6 - 16x^9y^5 - x^9y^4 \\
-3x^8y^{10} - 4x^8y^9 + 124x^8y^8 + 227x^8y^7 - 537x^8y^6 + 107x^8y^5 + 79x^8y^4 + 7x^8y^3 \\
+3x^7y^{10} + 66x^7y^9 + 227x^7y^8 - 504x^7y^7 - 1761x^7y^6 + 2132x^7y^5 + 16x^7y^4 - 140x^7y^3 - 12x^7y^2 \\
+x^6y^{10} - 34x^6y^9 - 537x^6y^8 - 1761x^6y^7 + 814x^6y^6 + 6231x^6y^5 - 4496x^6y^4 - 481x^6y^3 + 95x^6y^2 + 6x^6y \\
-16x^5y^9 + 107x^5y^8 + 2132x^5y^7 + 6231x^5y^6 - 1564x^5y^5 - 12565x^5y^4 + 5114x^5y^3 + 660x^5y^2 - 18x^5y \\
-x^4y^9 + 79x^4y^8 + 16x^4y^7 - 4496x^4y^6 - 12565x^4y^5 + 6034x^4y^4 + 15227x^4y^3 - 2941x^4y^2 - 273x^4y \\
+7x^3y^8 - 140x^3y^7 - 481x^3y^6 + 5114x^3y^5 + 15227x^3y^4 - 12842x^3y^3 - 10404x^3y^2 + 684x^3y \\
-12x^2y^7 + 95x^2y^6 + 660x^2y^5 - 2941x^2y^4 - 10404x^2y^3 + 12650x^2y^2 + 3057x^2y - 27x^2 \\
+6xy^6 - 18xy^5 - 273xy^4 + 684xy^3 + 3057xy^2 - 5076xy + 27x \\
-27y^2 + 27y + 324.
\end{gathered}
$$

We have $|E(H(P)) - E(P)| \leq 1$, so $D(x,y)$ can only vanish when N vanishes. We will show that $N(x,y) > 0$ except when $x = y = 1/\phi$. This shows that $E(H(P)) > E(P)$ unless P is in the regular class.

Since N is D_5-invariant, it suffices to prove that $N > 0$ on $(0,1/\phi)^2$. This follows from Lemma 4.3. Define

$$
N_1(x,y) = N\Big(\frac{1-x}{\phi}, \frac{1-y}{\phi}\Big). \tag{30}
$$

N_1 is positive on $(0,1)^2$ if and only N is positive on $(0,1/\phi)^2$. Listing out the lowest order nontrivial terms, we have

$$
N_1(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots,
$$

$$
a_{20} = a_{02} = 320 - 120\sqrt{5}, \qquad a_{11} = 460 - 220\sqrt{5}. \tag{31}
$$

Since $a_{00} = a_{10} = a_{01} = 0$ and $a_{11} < 0$, we have $A_{11} < 0$. This means that N_2 is not weak positive dominant. However we define

$$
N_2(x,y) = N_1(x,y) + \frac{a_{11}}{2}(x-y)^2. \tag{32}
$$

We have $N_2 \leq N_1$. So, if $N_2 > 0$ on $(0,1)^2$, then $N > 0$ on $(0,1/\phi)^2$. We check by direct calculation that N_2 is positive dominant. With respect to N_2 we have either $A_{ij} = 0$ or $A_{ij} \geq 550 - 230\sqrt{5}$. The total sum is $\sum a_{ij} = 324$. This completes the proof.

4.7 Sketch of a Second Proof

Here we sketch a second proof of Theorem 1.1 which seems less lucky and more robust.

A bounded open set $X \subset \boldsymbol{C}^n$ has a metric, called the *Kobayashi metric*, defined as follows. Let D be the open unit disk. Given $p \in X$ and a vector V in the tangent space T_pX, we consider all holomorphic maps $f : \Delta \to X$ such that $f(0) = p$ and $df(\boldsymbol{C})$ is spanned by V and iV. We then define

$$\|V\|_p = \inf_f \|df_p^{-1}(V)\|. \tag{33}$$

The norm on the right is just the Euclidean norm. We are taking the inf over all functions meeting the conditions above.

If $I : X_1 \to X_2$ is a biholomorphic map, then I is an isometry relative to the two Kobayashi metrics. Moreover, if $H : X \to X$ is any map, then H does not expand distances relative to the Kobayashi metric on X. Here is a concrete instance of this principle. We fix some point $p \in \boldsymbol{C}^n$ and let $Q(r)$ denote the cube of radius r about p. The following result is an easy exercise. We omit the proof.

Lemma 4.5 *Suppose that $H(Q(r)) \subset Q(s)$ for some $s < r$. Then H is a uniform contraction relative to the Kobayashi metric on $Q(r)$.*

By Lemma 4.3, the slightly larger square $(0, 2/3)^2$ contains a fundamental domain for D_5. In light of the fact that H commutes with the D_5 action, it suffices to prove the convergence for points in $(0, 2/3)^2$. Let $Q(r) \subset \boldsymbol{C}^2$ denote the cube of side-length r centered at $(1/3, 1/3)$. Note that we have the containment $(0, 2/3)^2 \subset Q(2/3)$.

One can check by direct numerical means – calculating on a sufficiently dense mesh and using bounds on the partial derivatives – that

$$H(Q(2/3)) \subset Q(2/3 - \epsilon), \tag{34}$$

for say $\epsilon = 1/1000$.

We equip $Q(2/3)$ with its Kobayashi metric. Combining Lemma 4.5 and Equation 34, we see that the restriction of H to $Q(2/3)$ is a uniform contraction in the Kobayashi metric. Hence, the iterates of H converge any starting point in $Q(2/3)$ exponentially fast to a unique fixed point. The fixed point must be the one we already know, namely $R = (1/\phi, 1/\phi)$.

5 The Action on Polygons

5.1 Comparison with Conformal Averaging

In [**Sch4**], I studied a projectively (or, equivalently, conformally) invariant iteration defined on convex polygons inscribed in the circle. The iteration is quite similar to the projective heat map. Suppose that P is a convex inscribed polygon with vertices $P_0, P_2, P_4, ... \in S^1$. Figure 4.1 shows how we can choose a point $Q_3 \in S^1$ in a projectively natural way from the circular arc bounded by P_2 and P_4, using the points P_0, P_2, P_6, P_6 as guides.

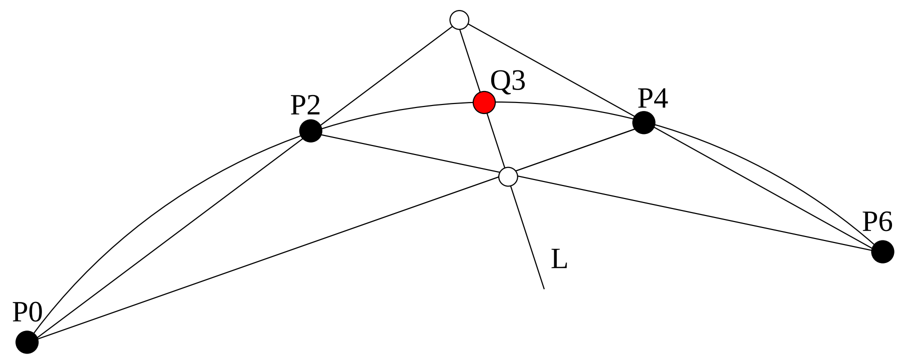

Figure 4.2: Choosing a point on a circular arc.

Equivalently, there is a projective transformation T, preserving S^1, so that

- $T(P_0) = (s, -s)$.
- $T(P_2) = (s, s)$.
- $T(Q_3) = (0, 1)$.
- $T(P_4) = (-s, s)$.
- $T(P_6) = (-s, -s)$.

Here $s = \sqrt{2}/2$.

Using this construction, we build a map $\widehat{H}$ on the space of inscribed convex polygons. In [**Sch4**], I proved a version of Conjecture 1.2, Statement 1. Let $\mathcal{IC}_n$ denote the space of projective equivalence classes of convex n-gons which can be inscribed in the unit circle.

Theorem 5.1 *The projective energy E is an increasing quantity for $\widehat{H}$ on $\mathcal{IC}_n$. That is, $E(H(P)) \geq E(P)$ for all $P \in \mathcal{IC}_n$ with equality if and only if P is projectively regular.*

As a fairly immediate corollary, we get

Corollary 5.2 *Suppose $n \geq 5$. For any $x \in \mathcal{IC}_n$, the sequence $\{\widehat{H}^n(x)\}$ converges to the projectively regular class.*

One should contrast Theorem 5.1 with Theorem 1.3, which says that no result like Theorem 5.1 is possible with the projective heat map H in place of $\widehat{H}$.

A more obvious difference between the maps $\widehat{H}$ and H is that $\widehat{H}$ is only defined for convex inscribed polygons whereas H is defined for any generic polygon. Without the cyclic ordering on the points given by the convexity, it seems difficult to make the construction. Referring to Figure 4.1, the problem is that the line L intersects the circle in two points, and one must make a choice between these two points. One approach might be to give a separate rule for every topological possibility, but this seems a bit artificial.

Despite its limited definition, one advantage that $\widehat{H}$ has over H is that $\widehat{H}$ keeps the polygons about the same size whereas H shrinks them. For this reason, one can make meaningful statements about how $\widehat{H}$ acts on polygons, whereas most of the meaningful statements about H have to do its action on equivalence classes of polygons.

5.2 The Pentagram Map

Figure 2.5 shows the action of the pentagram map on a convex hexagon. The map in general is similar. As we mentioned in the introduction, there is now a fairly extensive literature on the pentagram map. See [**OST2**] or [**S5**] for an up-to-date list of references. We denote the pentagram map by Φ.

The pentagram map is now known to be a discrete, completely integrable system. See [**OST1**], [**OST2**], and [**Sol**. This is to say that Φ has an invariant Poisson structure and sufficiently many invariant functions which Poisson-commute. Concretely, this means that $\mathcal{C}_n$ has a singular foliation by tori which is invariant under Φ and each torus in the foliation has a natural flat structure. The orbit of almost every point $x \in \mathcal{C}_n$ is contained in a finite union T_x of tori, and the restriction of Φ to T_x is a translation relative to the flat structure.

The pentagram map behaves naturally with respect to projective duality. Let Φ^* be the dual to the pentagram map, defined just as we defined H^* in §3.2. Then, up to choosing the correct labeling convention, we have

$$\Phi^* = \Phi^{-1}. \tag{35}$$

The integrals for the pentagram map come in two families, $O_{n,a}$ and $E_{n,a}$ for $a = 1, ..., [n/2], n$. Here $[n/2]$ is the floor of $n/2$. These invariants are related by the equations

$$O_{n,a}(P) = E_{n,a}(P^*), \qquad O_{n,a}(\Phi P) = E_{n,a}(P). \tag{36}$$

The same equations hold with the roles of O and E reversed. Technically, these functions are invariants for Φ^2 rather than Φ. The self-dual quantities, such as $O_{n_a}E_{n,a}$ are invariants for Φ. The quantity $O_{n,n}E_{n,n}$ is the projective energy discussed in the previous chapters. (There we set $O_n = O_{n,n}$ and $E_n = E_{n,n}$ for convenience.)

The projective heat map is really built out of the pentagram map. Given a polygon P, there is a natural correspondence between the vertices of $\Phi(P)$ and the vertices of $\Phi^{-1}(P)$. We get the polygon $H(P)$ by taking the lines through corresponding vertices of $\Phi(P)$ and $\Phi^{-1}(P)$ and intersecting these lines with P. The dual map H^* has a similar interpretation.

The pentagram map and the projective heat map are both projectively natural and defined for generic polygons over any field. In this formal sense, the projective heat map seems most closely related to the pentagram map.

One way that the two maps differ is that Equation 35 fails for H. Indeed, the map H is generally many-to-one, and hence not invertible. The most significant difference between H and Φ is that Φ is integrable and H is not. The maps are extremely different dynamically. On $\mathcal{C}_n$, the map Φ exhibits quasiperiodic motion whereas H seems to attract everything to the projectively regular class. Because H and Φ are so different dynamically, it does not seem worthwhile to make further dynamical comparisons. There is something different we can do.

Since H and Φ have the same domains and naturality properties, it makes to consider the semigroup

$$\mathcal{S} = \{H, H^*, \Phi, \Phi^*\} \tag{37}$$

acting on $\mathcal{P}_n$ and $\mathcal{C}_n$. By considering the action of various words in $\mathcal{S}$, I was hoping to get a feel for how H and Φ interact, as opposed to how they directly compare.

5.3 Some Conjectures

Let R_n denote the projectively regular class.

The most interesting element I've found so far in $\mathcal{S}$ is

$$\Psi = H^2\Phi. \tag{38}$$

Let $[\Psi]$ denote the differential $d\Psi$ evaluated at R_n, the projectively regular class. It seems that R_n is a global attractor for Ψ when $n \leq 11$ but not when $n \geq 12$.

When $n \geq 18$, it seems that there is an invariant torus $T_n \subset \mathcal{P}_n$ which is a global attractor for Ψ. The restriction of Ψ to T_n looks like it is conjugation to a translation. Thus Ψ seems to exhibit both attracting and integrable bahavior for $n \geq 18$. I should say that I don't have complete confidence in my experiments.

Some computer experimentation (but not really enough) leads me to the following conjecture.

Conjecture 5.3 *Let $\mathcal{T}$ denote the semigroup generated by H and Φ. Then any element of $\mathcal{T}$ acts on $\mathcal{P}_n$ so as to have R_n as gloal attractor.*

Φ is the identity on $\mathcal{P}_5$. Hence, Theorem 1.1 implies this conjecture in a trivial way, for $n = 5$. Here is another conjecture which is inspired by some numerical evidence.

Conjecture 5.4 *Let $P \in \mathcal{C}_6$ be arbitrary. Let T be any of the following maps*

- $HH^*\Phi^{2k}$
- $H^*H(\Phi^*)^{2k}$
- $HH^*(\Phi^*)^{2k}$
- $H^*H(\Phi)^{2k}$.

Then $\{T^m(P)\}$ converges to the class of a hexagon with 6-fold dihedral symmetry.

Conjecture 5.4 cannot be improved because these hexagons with 6-fold symmetry are fixed by HH^* and H^*H, as mentioned in the proof of Theorem 1.3. I might be able to prove Conjecture 5.4, at least for several of the maps, along the lines of Theorem 1.1, using a relative version of the contraction principle discussed in §4.7. However, I'm not sure if the calculation is feasible.

I looked at 4 of these maps in some more detail (though not too much more).

Conjecture 5.5 *Let T be any of the maps in Conjecture 5.4 with $k = 1$, and suppose n is even. Then, for any $P \in \mathcal{C}_n$, the sequence $\{T^m(P)\}$ converges to the class of a hexagon with n-fold dihedral symmetry.*

When n is odd the maps in Conjecture 5.5 seem to have the regular class as a global attractor. Thus, for instance, the map $HH^*\Phi^2$ seems closer to heat flow than HH^*, which has many other fixed points besides the regular class.

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