Quasiperiodic Motion for the Pentagram Map

Valentin Ovsienko, Richard Schwartz, and Sergei Tabachnikov

December 16, 2008

1 Introduction

The pentagram map, T, is a natural operation one can perform on polygons. See [**S1**], [**S2**] and [**OST**] for the history of this map and additional references. Though this map can be defined for an essentially arbitrary polygon in an essentially arbitrary field, it is easiest to describe the map for convex polygons contained in $\boldsymbol{R}^2$. Given such an n-gon P, the corresponding n-gon $T(P)$ is the convex hull of the intersection points of consequtive shortest diagonals of P. Figure 1 shows two examples.

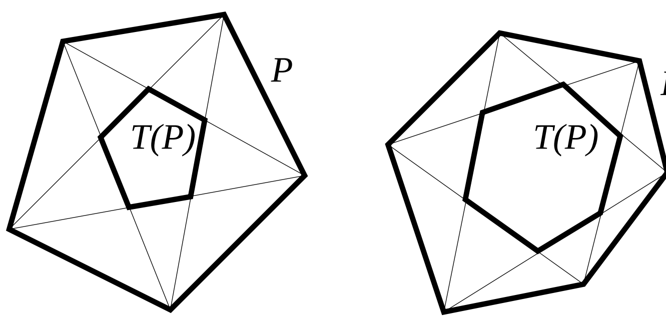

Figure 1: The pentagram map defined on a pentagon and a hexagon.

Thinking of $\boldsymbol{R}^2$ as a natural subset of the projective plane $\boldsymbol{RP}^2$, we observe that the pentagram map commutes with projective transformations. That is, $\phi(T(P)) = T(\phi(P))$, for any projective transformation ϕ. Indeed, the pentagram map induces a self-diffeomorphism of the space $\mathcal{C}_n$ of convex n-gons modulo projective transformations. We again denote this map by T.

It turns out that T is the identity map on $\mathcal{C}_5$ and an involution on $\mathcal{C}_6$. For $n \geq 7$, the map T exhibits quasi-periodic properties. Experimentally, the orbits of T exhibit the kind of quasiperiodic motion associated to completely integrable systems. It seems that T preserves a certain foliation of $\mathcal{C}_n$ by roughly half-dimensional tori, and the action of T on each torus is conjugate to a rotation. Our recent paper [**OST**] very nearly proves this result.

Rather than work directly with $\mathcal{C}_n$, we work with a slightly larger space, which we call $\mathcal{P}_n$. The space $\mathcal{P}_n$ is the space of *twisted n-gons* modulo projective equivalence. (We give the formal definition in §2.1 below.) We introduce coordinates that identify $\mathcal{P}_n$ with an open subset of $\boldsymbol{R}^{2n}$. Using these coordinates, we show that the pentagram map preserves a Poisson structure that is completely integrable in the sense of Arnold–Liouville, [**A**], and we give an explicit and complete list of invariants (or integrals) for the map. This is the algebraic part of our theory.

The space $\mathcal{C}_n$ is naturally a subspace of $\mathcal{P}_n$, and our algebraic results say something (but not quite enough) about the action of the pentagram map on $\mathcal{C}_n$. There are still some details about how the Poisson structure and the invariants restrict to $\mathcal{C}_n$ that we have yet to work out. To get a crisp geometric result, we work with a related space, which we describe next.

We say that a *universally convex* twisted n-polygon is a map $\phi : \boldsymbol{Z} \to \boldsymbol{R}^2$ such that $\phi(\boldsymbol{Z})$ is convex and contained in the positive quadrant. We require that

$$\phi(k+n) = M \circ \phi(k); \qquad \forall k \in \boldsymbol{Z}. \tag{1}$$

Here $M : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a linear transformation having the form

$$M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; \qquad a < 1 < b \tag{2}$$

The image of ϕ looks somewhat like a "polygonal hyperbola". We say that two universally convex twisted n-gons ϕ_1 and ϕ_2 are *equivalent* if there is a positive diagonal matrix μ such that $\mu \circ \phi_1 = \phi_2$. Let $\mathcal{U}_n$ denote the space of universally convex twisted n-gons modulo equivalence. It turns out that $\mathcal{U}_n$ is a pentagram-invariant and open subset of $\mathcal{P}_n$. Here is our main geometric result.

Theorem 1.1 *Almost every pont of $\mathcal{U}_n$ lies on a smooth torus that has a T-invariant affine structure. Hence, the orbit of almost every universally convex n-gon undergoes quasi-periodic motion under the pentagram map.*

In this note we will sketch the main ideas in the proof of Theorem 1.1. We refer the reader to [**OST**] for more results and details, as well as a discussion of the context behind the integrability we establish.

2 Sketch of the Proof

2.1 Coordinates

A twisted n-gon is a map $\phi : \boldsymbol{Z} \to \boldsymbol{RP}^2$ such that

$$\phi(n+k) = M \circ \phi(k) \tag{3}$$

for some projective transformation M and all k. We let $v_i = \phi(i)$. Thus, the vertices of our twisted polygon are naturally $...v_{i-1}, v_i, v_{i+1}, ...$ Our standing assumption is that v_{i-1}, v_i, v_{i+1} are in general position for all i, but sometimes this assumption alone will not be sufficient for our constructions.

Recall that the *cross ratio* of 4 collinear points in $\boldsymbol{RP}^2$ is given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)\,(t_3 - t_4)}{(t_1 - t_3)\,(t_2 - t_4)}. \tag{4}$$

To get this expression, we use a projective transformation to identify the line containing the points with $\boldsymbol{R}$; the result is independent of the choice. We use the cross ratio to construct coordinates on the space of twisted polygons. We associate to every vertex v_i two numbers:

$$\begin{aligned} x_i &= [v_{i-2},\, v_{i-1},\, ((v_{i-2}, v_{i-1}) \cap (v_i, v_{i+1}))\,,\, ((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))] \\ y_i &= [((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))\,,\, ((v_{i-1}, v_i) \cap (v_{i+1}, v_{i+2}))\,,\, v_{i+1},\, v_{i+2}] \end{aligned} \tag{5}$$

called the left and right corner cross-ratios. We often call our coordinates the *corner invariants*.

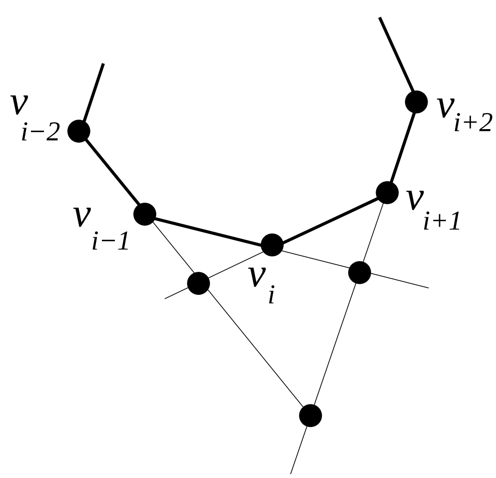

Figure 2: Points involved in the definition of the invariants.

This construction is invariant under projective transformations, and thus gives us coordinates on the space $\mathcal{P}_n$. At generic points, $\mathcal{P}_n$ is locally diffeomorphic to $\boldsymbol{R}^{2n}$.

To save words later, we say now that we will work with generic elements of $\mathcal{P}_n$, so that all constructions are well-defined. Let $\phi^* = T(\phi)$ be the image of ϕ under the pentagram map. We choose the labelling scheme shown in Figure 3. The black dots represent ϕ and the white ones represent ϕ^*.

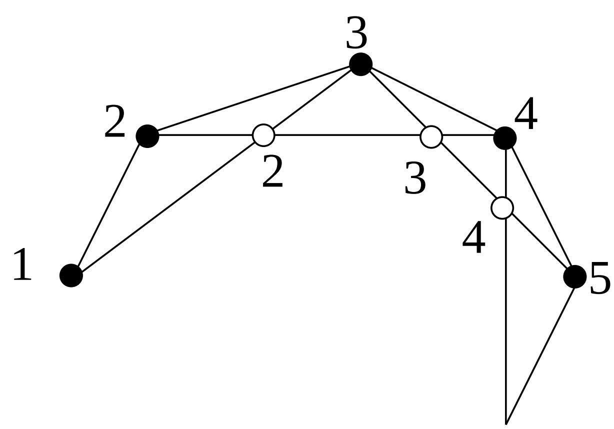

Figure 3: The labelling scheme.

Now we describe the pentagram map in coordinates. Suppose the coordinates for ϕ are $x_1, y_1, ...$ and the coordinates for $\phi^* = T(\phi)$ are $x_1^*, y_1^*, ...$ then

$$x_i^* = x_i \,\frac{1 - x_{i-1}\, y_{i-1}}{1 - x_{i+1}\, y_{i+1}}, \qquad y_i^* = y_{i+1}\, \frac{1 - x_{i+2}\, y_{i+2}}{1 - x_i\, y_i}, \tag{6}$$

Equation 6 has two immediate corollaroes. First, there is an interesting scaling symmetry of the pentagram map. We have a *rescaling operation* on $\boldsymbol{R}^{2n}$, given by the expression

$$R_t: \quad (x_1, y_1, ..., x_n, y_n) \to (tx_1, t^{-1}y_1, ..., tx_n, t^{-1}y_n). \tag{7}$$

Corollary 2.1 *The pentagram map commutes with the rescaling operation.*

Second, the formula for the pentagram map exhibits rather quickly some invariants of the pentagram map. When n is odd, define

$$O_n = \prod_{i=1}^{n} x_i; \qquad E_n = \prod_{i=1}^{n} y_i \tag{8}$$

When n is even, define

$$O_{n/2} = \prod_{i \text{ even}} x_i + \prod_{i \text{ odd}} x_i. \qquad E_{n/2} = \prod_{i \text{ even}} y_i + \prod_{i \text{ odd}} y_i. \tag{9}$$

The products in this last equation run from 1 to n.

Corollary 2.2 *When n is odd, the functions O_n and E_n are invariant under the pentagram map. When n is even, the functions $O_{n/2}$ and $E_{n/2}$ are also invariant under the pentagram map.*

2.2 The Monodromy Invariants

In this section we describe the invariants of the pentagram map. We call them the *monodromy invariants*. As above, let ϕ be a twisted n-gon with invariants $x_1, y_1, \ldots$. Let M be the monodromy of ϕ. We lift M to an element of $GL_3(\boldsymbol{R})$. By slightly abusing notation, we also denote this matrix by M. The two quantities

$$\Omega_1 = \frac{\text{trace}^3(M)}{\det(M)}; \qquad \Omega_2 = \frac{\text{trace}^3(M^{-1})}{\det(M^{-1})}; \tag{10}$$

enjoy 3 properties.

- Ω_1 and Ω_2 are independent of the lift of M.
- Ω_1 and Ω_2 only depend on the conjugacy class of M.
- Ω_1 and Ω_2 are rational functions in the corner invariants.

We define

$$\widetilde{\Omega}_1 = O_n^2 E_n \Omega_1; \qquad \widetilde{\Omega}_2 = O_n E_n^2 \Omega_2. \tag{11}$$

In [**S3**] (and again in [**OST**]) it is shown that $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are polynomials in the corner invariants. Since the pentagram map preserves the monodromy, and O_n and E_n are invariants, the two functions $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are also invariants.

We say that a polynomial in the corner invariants has *weight k* if we have the following equation

$$R_t^*(P) = t^k P. \tag{12}$$

here R_t^* denotes the natural operation on polynomials defined by the rescaling operation above. For instance, O_n has weight n and E_n has weight $-n$. In [**S3**] it shown that

$$\widetilde{\Omega}_1 = \sum_{k=1}^{[n/2]} O_k; \qquad \widetilde{\Omega}_2 = \sum_{k=1}^{[n/2]} E_k \tag{13}$$

where O_k has weight k and E_k has weight $-k$. Since the pentagram map commutes with the rescaling operation and preserves $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$, it also preserves their "weighted homogeneous parts". That is, the functions $O_1, E_1, O_2, E_2, ...$ are also invariants of the pentagram map. These are the monodromy invariants. They are all nontrivial polynomials. In [**S3**] it is shown that the monodromy invariants are algebraically independent.

2.3 The Poisson Bracket

In [**OST**] we introduce the Poisson bracket on $\mathcal{P}_n$. Let C_n^∞ denote the algebra of smooth functions on $\boldsymbol{R}^{2n}$. A *Poisson structure* on C_n^∞ is a map

$$\{\ ,\ \} : C_n^\infty \times C_n^\infty \to C_n^\infty \tag{14}$$

that obeys the following axioms.

1. Antisymmetry: $\{f, g\} = -\{g, f\}$
2. Linearity: $\{af_1 + f_2, g\} = a\{f_1, g\} + \{f_2, g\}$.
3. Leibniz Identity: $\{f, g_1 g_2\} = g_1\{f, g_2\} + g_2\{f, g_1\}$.
4. Jacobi Identity: $\Sigma\{f_1, \{f_2, f_3\}\} = 0$.

Here Σ denotes the cyclic sum.

We define a following Poisson bracket on the coordinate functions of $\boldsymbol{R}^{2n}$.

$$\{x_i, x_{i\pm1}\} = \mp x_i\, x_{i+1}, \qquad \{y_i, y_{i\pm1}\} = \pm y_i\, y_{i+1} \tag{15}$$

All other brackets not explicitly mentioned above vanish. Once we have the definition on the coordinate functions, we use linearity and the Liebniz rule to extend to all rational functions. An easy exercise shows that the Jacobi identity holds.

Two functions f and g are said to *Poisson commute* if $\{f, g\} = 0$. A function f is said to be a *Casimir* (relative to the Poisson structure) if f Poisson commutes with all other functions.

The *corank* of a Poisson bracket on a smooth manifold is the codimension of the generic symplectic leaves. These symplectic leaves can be locally described as levels $F_i =$ const of the Casimir functions. See [**W**] for the details.

Now that we have these basic definitions in place, we can state the main lemmas we establish in [**OST**] concerning our Poisson bracket. We establish 4 things.

1. The Poisson bracket is invariant with respect to the Pentagram map.
2. The monodromy invariants Poisson commute.
3. The invariants in Equations 8 and (in the even case) 9 are Casimirs.
4. The Poisson bracket has corank 2 if n if odd and corank 4 if n is even.

We now consider the case when n is odd. The even case is similar. On the space $\mathcal{P}_n$ we have a generically defined and T-invariant Poisson bracked that is invariant under the pentagram map. This bracket has co-rank 2, and the generic level set of the Casimir functions has dimension $4[n/2] = 2n-2$. On the other hand, after we exclude the two Casimirs, we have $2[n/2] = n-1$ algebraically independent invariants that Poisson commute with each other. This gives us the classical Liouville-Arnold complete integrability.

2.4 The End of the Proof

Now we specialize our algebraic result to the space $\mathcal{U}_n$ of universally convex twisted n-gons. We check that $\mathcal{U}_n$ is an open and invariant subset of $\mathcal{P}_n$. The invariance is pretty clear. The openness result derives from 3 facts.

1. Local convexity is stable under perturbation.
2. The linear transformations in Equation 2 extend to projective transformations whose type is stable under small perturbations.
3. A locally convex twisted polygon that has the kind of hyperbolic monodromy given in Equation 2 is actually globally convex.

As a final ingredient in our proof, we show that the leaves of $\mathcal{U}_n$, namely the level sets of the monodromy invariants, are compact. We don't need to consider all the invariants; we just show in a direct way that the level sets of E_n and O_n together are compact.

The rest of the proof is the usual application of Sard's theorem and the definition of integrability. We explain the main idea in the odd case. The space $\mathcal{U}_n$ is locally diffeomorphic to $\boldsymbol{R}^{2n}$, and foliated by leaves which generically are smooth compact symplectic manifolds of dimension $2n-2$. A generic point in a generic leaf lies on an $(n-1)$ dimensional smooth compact manifold, the level set of our monodromy invariants. On a generic leaf, the symplectic gradients of the monodromy functions are linearly independent at each point of the leaf.

The $n-1$ symplectic gradients of the monodrony invariants give a natural basis of the tangent space at each point of our generic leaf. This basis is invariant under the pentagram map, and also under the Hamiltonian flows determined by the invariants. This gives us a smooth compact $n-1$ manifold, admitting $n-1$ commuting flows that preserve a natural affine structure. From here, we see that the leaf must be a torus. The pentagram map preserves the canonical basis of the torus at each point, and hence acts as a translation. This is the quasi-periodic motion of Theorem 1.1.

3 References

[**A**] V. I. Arnold, Mathematical Methods of Classical Mechanics, *Graduate Texts in Mathematics* **60**, Springer-Verlag, New York, 1989

[**OST**] V. Ovsienko, S. Tabachnikov, R. Schwartz, *The Pentagram Map: A Discrete Integrable System*, submitted (2008)

[**S1**] R. Schwartz, *The Pentagram Map*, Experimental Math **1** (1992) pp 71-81.

[**S2**] R. Schwartz, *Discrete Monodomy, Pentagrams, and the Method of Condensation* J. of Fixed Point Theorem and Appl. **3** (2008) pp 379-409

[**W**] A. Weinstein, *The local structure of Poisson Manifolds*, J. Diff Geometry **18** (1983) pp 523-557