Elementary Surprises in Projective Geometry

Richard Evan Schwartz* and Serge Tabachnikov†

The classical theorems in projective geometry involve constructions based on points and straight lines. A general feature of these theorems is that a surprising coincidence awaits the reader who makes the construction. A classical example of this is Pappus's theorem. One starts with 6 points, 3 of which are contained on one line and 3 of which are contained on another. Drawing the additional lines shown in Figure 1, one sees that the 3 middle (black) points are also contained on a line.

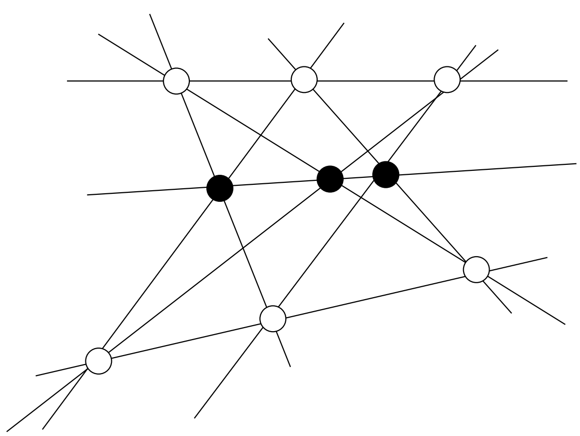

Figure 1: Pappus's Theorem.

Pappus's Theorem goes back about 2300 years. In 1639, Blaise Pascal discovered a generalization of Pappus's Theorem. In Pascal's Theorem, the 6 white points are contained in a conic section, as shown on the left hand side of Figure 2.

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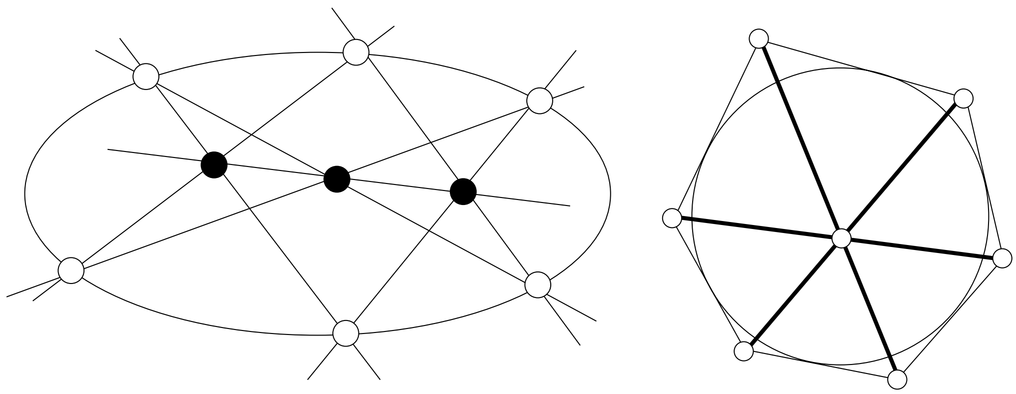

Figure 2: Pascal's Theorem and Brianҫon's Theorem

One recovers Pappus's Theorem as a kind of limit, as the conic section stretches out and degenerates into a pair of straight lines.

Another closely related theorem is Brianҫon's Theorem. This time, the 6 white points are the vertices of a hexagon that is circumscribed about a conic section, as shown on the right hand side of Figure 2, and the surprise is that the 3 thickly drawn diagonals intersect in a point. Though Brianҫon discovered this result about 200 years after Pascal's theorem, the two results are in fact equivalent for the well-known reason we will discuss below.

The purpose of this article is to discuss some apparently new theorems in projective geometry that are similar in spirit to Pascal's Theorem and Brianҫon's Theorem. One can think of all the results we discuss as statements about lines and points in the ordinary Euclidean plane, but setting the theorems in the *projective plane* enhances them.

The Basics of Projective Geometry: Recall that the projective plane $\boldsymbol{P}$ is defined as the space of lines through the origin in $\boldsymbol{R}^3$. A point in $\boldsymbol{P}$ can be described by *homogeneous coordinates* $(x : y : z)$, not all zero, corresponding to the line containing the vector (x, y, z). Of course, the two triples $(x : y : z)$ and $(ax : ay : az)$ describe the same point in $\boldsymbol{P}$ as long as $a \neq 0$. One says that $\boldsymbol{P}$ is the *projectivization* of $\boldsymbol{R}^3$.

A *line* in the projective plane is defined as a set of lines through the origin in $\boldsymbol{R}^3$ that lie in a plane through the origin. Any linear isomorphism of $\boldsymbol{R}^3$ – i.e., multiplication by an invertible 3×3 matrix – permutes the lines and planes through the origin. In this way, a linear isomorphism induces a mapping of $\boldsymbol{P}$ that carries lines to lines. These maps are called *projective transformations*.

One way to define a (non-degenerate) conic section in $\boldsymbol{P}$ is to say that

- The set of points in $\boldsymbol{P}$ of the form $(x : y : z)$ such that $z^2 = x^2 + y^2 \neq 0$ is a conic section.
- Any other conic section is the image of the one we just described under a projective transformation.

One frequently identifies $\boldsymbol{R}^2$ as the subset of $\boldsymbol{P}$ corresponding to points $(x : y : 1)$. We will simply write $\boldsymbol{R}^2 \subset \boldsymbol{P}$. The ordinary lines in $\boldsymbol{R}^2$ are subsets of lines in $\boldsymbol{P}$. The conic sections intersect $\boldsymbol{R}^2$ either in ellipses, hyperbolas, and parabolas. One of the beautiful things about the projective geometry is that these three kinds of curves are *the same* from the point of view of the projective plane and its symmetries.

The *dual plane* $\boldsymbol{P}^*$ is defined to be the set of planes through the origin in $\boldsymbol{R}^3$. Every such plane is the kernel of a linear function on $\boldsymbol{R}^3$, and this linear function is determined by the plane up to a non-zero factor. Hence $\boldsymbol{P}^*$ is the projectivization of the dual space $(\boldsymbol{R}^3)^*$. If one wishes, one can identify $\boldsymbol{R}^3$ with $(\boldsymbol{R}^3)^*$ using the scalar product. One can also think of $\boldsymbol{P}^*$ as the space of lines in $\boldsymbol{P}$. A projective transformation $\boldsymbol{P} \to \boldsymbol{P}^*$ is induced by a linear map $\boldsymbol{R}^3 \to (\boldsymbol{R}^3)^*$.

Given a point v in $\boldsymbol{P}$, the set $v^\perp$ of linear functions on $\boldsymbol{R}^3$, that vanish at v, determine a line in $\boldsymbol{P}^*$. The correspondence $v \mapsto v^\perp$ carries collinear points to concurrent lines; it is called the *projective duality*. Projective duality takes points of $\boldsymbol{P}$ to lines of $\boldsymbol{P}^*$, and lines of $\boldsymbol{P}$ to points of $\boldsymbol{P}^*$. Of course, the same construction works in the opposite direction, from $\boldsymbol{P}^*$ to $\boldsymbol{P}$. Projective duality is an involution: applied twice, it yields the identity map.

Projective duality extends to smooth curves: the 1-parameter family of the tangent lines to a curve γ in $\boldsymbol{P}$ is a 1-parameter family of points in $\boldsymbol{P}^*$, the dual curve γ^*. The curve dual to a conic section is again a conic section. Thus projective duality carries the vertices of a polygon inscribed in a conic to the lines extending the edges of a polygon circumscribed about a conic.

Projective duality takes an instance of Pascal's Theorem to an instance of Brianҫon's Theorem, and vice versa. This becomes clear if one looks at the objects involved. The input of Pascal's theorem is an inscribed hexagon and the output is 3 collinear points. The input of Brianҫon's theorem is a superscribed hexagon and the output is 3 coincident lines.

Polygons: Like Pascal's Theorem and Brianҫon's Theorem, our results all involve polygons. A polygon P in $\boldsymbol{P}$ is a cyclically ordered collection $\{p_1, ..., p_n\}$ of points, its vertices. A polygon has sides: the cyclically ordered collection $\{l_1, ..., l_n\}$ of lines in $\boldsymbol{P}$ where $l_i = \overline{p_i p_{i+1}}$ for all i. Of course, the indices are taken mod n. The *dual polygon* P^* is the polygon in $\boldsymbol{P}^*$ whose vertices are $\{l_1, ..., l_n\}$; the sides of the dual polygon are $\{p_1, ..., p_n\}$ (considered as lines in $\boldsymbol{P}^*$). The polygon, dual to the dual, is the original one: $(P^*)^* = P$.

Let $\mathcal{P}_n$ and $\mathcal{P}_n^*$ denote the sets of n-gons in $\boldsymbol{P}$ and $\boldsymbol{P}^*$, respectively. There is a natural map $T_k : \mathcal{P}_n \to \mathcal{P}_n^*$. Given an n-gon $P = \{p_1, ..., p_n\}$, we define $T_k(P)$ as

$$\{\overline{p_1 p_{k+1}}, \overline{p_2 p_{k+2}}, \ldots \overline{p_n, p_{k+n}}\}.$$

That is, the vertices of $T_k(P)$ are the consecutive k-diagonals of P. The map T_k is an *involution*, meaning that T_k^2 is the identity map. When $k = 1$, the map T_1 carries a polygon to the dual one.

When $a \neq b$, the map $T_{ab} = T_a \circ T_b$ carries $\mathcal{P}_n$ to $\mathcal{P}_n$ and $\mathcal{P}_n^*$ to $\mathcal{P}_n^*$. We have studied the dynamics of the *pentagram map* T_{12} in detail in [2, 3, 4, 5, 6], and the configuration theorems we present here are a byproduct of that study. We extend the notation: $T_{abc} = T_a \circ T_b \circ T_c$, and so on.

Now we are ready to present our configuration theorems.

The Theorems: To save words, we say that an *inscribed polygon* is a polygon whose vertices are contained in a conic section. Likewise, we say that a *circumscribed polygon* is a polygon whose sides are tangent to a conic. The projective duality carries inscribed polygons to circumscribed ones and vice versa. We say that two polygons, P in $\boldsymbol{P}$ and Q in $\boldsymbol{P}^*$, are *equivalent* if there is a projective transformation $\boldsymbol{P} \to \boldsymbol{P}^*$ that takes P to Q. In this case, we write $P \sim Q$.

Theorem 1 *The following is true.*

- *If P is an inscribed 6-gon, then $P \sim T_2(P)$.*
- *If P is an inscribed 7-gon, then $P \sim T_{212}(P)$.*
- *If P is an inscribed 8-gon, then $P \sim T_{21212}(P)$.*

Figure 3 illustrates[1] the third of these results. The outer octagon P is inscribed in a conic and the innermost octagon $T_{121212}(P) = (T_{21212}(P))^*$ is circumscribed about a conic.

[1]Our Java applet does a much better job illustrating these results. To play with it online, see http://www.math.brown.edu/∼res/Java/Special/Main.html.

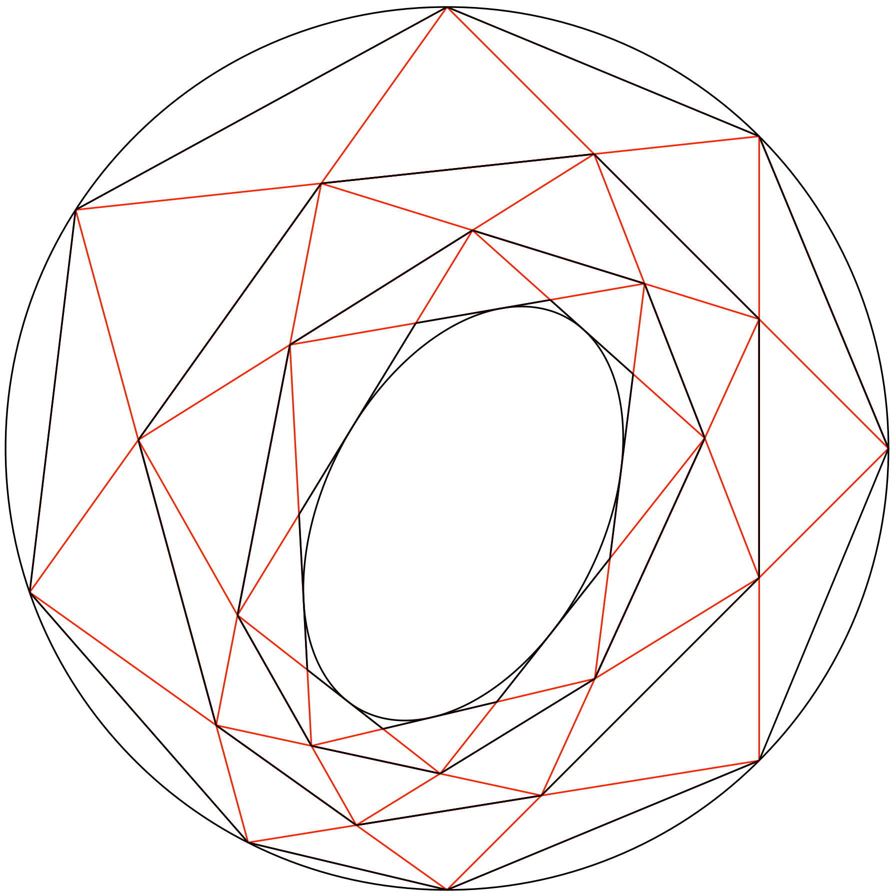

Figure 3: If P is an inscribed octagon then $P \sim T_{21212}(P)$.

The reader might wonder if our three results are the beginning of an infinite pattern. Alas, it is not true that P and $T_{2121212}(P)$ are equivalent when P is in inscribed 9-gon, and the predicted result fails for larger n as well. However, we do have a similar result for $n = 9, 12$.

Theorem 2 *If P is a circumscribed 9-gon, then $P \sim T_{313}(P)$.*

Theorem 3 *If P is an inscribed 12-gon, then $P \sim T_{3434343}(P)$.*

In our last collection of results, the conclusion is weaker. The "inner polygon" is not necessarily equivalent to the initial "outer polygon", but nonetheless it is always circumscribed.

Theorem 4 *The following is true.*

- *If P is an inscribed 8-gon, then $T_3(P)$ is circumscribed.*
- *If P is an inscribed 10-gon, then $T_{313}(P)$ is circumscribed.*
- *(*) If P is an inscribed 12-gon, then $T_{31313}(P)$ is circumscribed.*

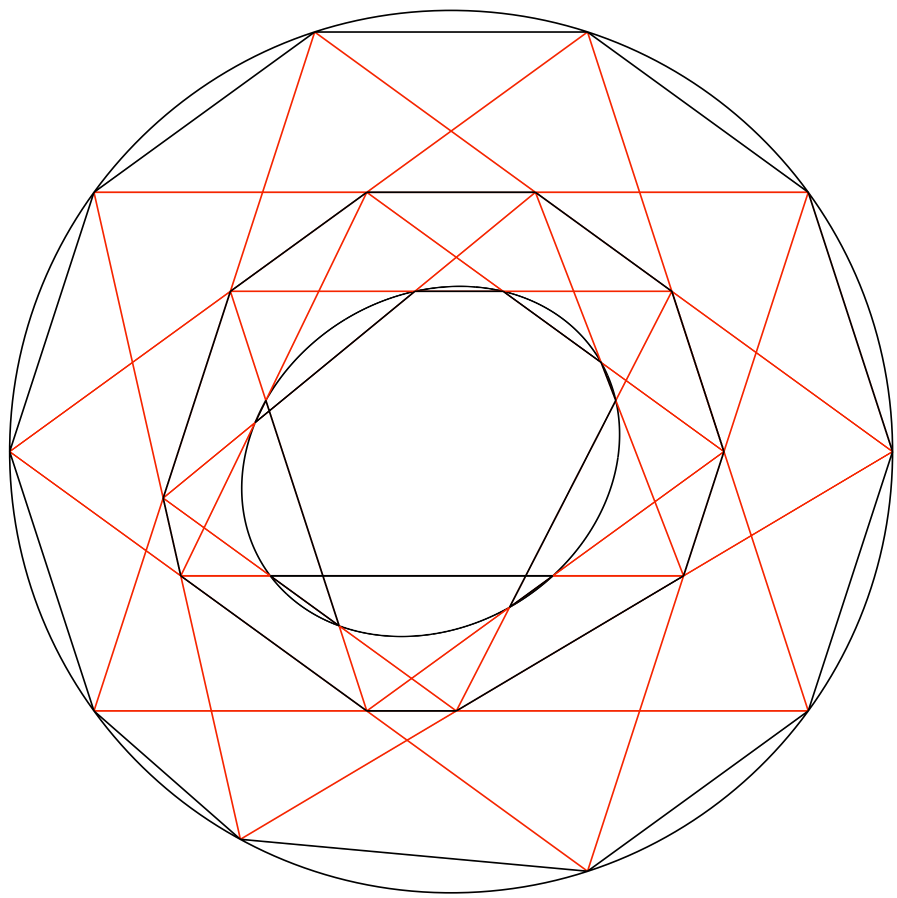

Figure 4: If P is an inscribed decagon then $T_{1313}(P)$ is also inscribed.

We have starred the third result because we don't yet have a proof for this one. Figure 4 illustrates the second of these results. Looking carefully, we see that $T_{1313}(P)$ is not even convex. So, even though $T_{1313}(P)$ is inscribed, it is not projectively equivalent to P nor to its dual P^*. One might wonder if this result is part of an infinite pattern, but once again the pattern stops after $n = 12$.

Discovery and Proof: We discovered these results through computer experimentation. We have been studying the dynamics of the pentagram map T_{12} on general polygons, and we asked ourselves whether we could expect any special relations when the intial polygon was either inscribed or circumscribed. We initially found the 7-gon result mentioned above. Then V. Zakharevich, a participant of the Penn State REU program in 2009, found Theorem 2. Encouraged by this good luck, we made a more extensive computer search that turned up the remaining results. We think that the list above is exhaustive, in the sense that there aren't any other surprises to be found by applying some combination of diagonal maps to inscribed or superscribed polygons. In particular, we don't think that surprises like the ones we found exist for N-gons with $N > 12$.

The reader might wonder how we prove the results above. In several of the cases, we found some nice geometric proofs which we will describe in a longer version of this article. With one exception, we found uninspiring algebraic proofs for the remaining cases. Here is a brief description of these algebraic proofs. First, we use symmetries of the projective plane to reduce to the case when the vertices of P lie on the parabola $y = x^2$. We represent vertices of P in homogeneous coordinates in the form $(t : t^2 : 1)$. Computing the maps $T_k(P)$ involves taking some cross products of the vectors $(t, t^2, 1)$ in $\boldsymbol{R}^3$. At the end of the construction, our claims about the final polygon boil down to equalities between determinants of various 3×3 matrices made from the vectors we generate. We then check these identities symbolically.

This approach has served to prove all but one of our results: the starred case of Theorem 4. The intensive symbolic manipulation required for this case is currently beyond what we can manage in Mathematica. The polynomials involved are absolutely huge. Naturally, we hope for some clever cancellations that we haven't yet been able to find.

We hope to find nice proofs for all the results above, but so far this has eluded us. Perhaps the interested reader will be inspired to look for nice proofs. We also hope that these results point out some of the beauty of the dynamical systems defined by these iterated diagonal maps.

Additional Remarks: In this concluding section, we relate our results to some other classical constructions in projective geometry, and also give some additional perspective on them.

1). Let us say a few words about pentagons. The following is true:

- Every pentagon is inscribed in a conic and circumscribed about a conic.
- Every pentagon is projectively equivalent to its dual.
- The pentagram map is the identity for every pentagon: $T_{12}(P) = P$.

We do not want to deprive the reader from the pleasure of discovering proofs to the latter two claims (in case of difficulty, see [1] and [2]). Therefore one may add the following to Theorem 1: *If P is a 5-gon, then $P \sim T_2(P)$.*

Related to the second item above, is the notion of *self-polar* spherical polygon. Let $p_1, \ldots, p_5$ be the vertices of a spherical pentagon. The pentagon is called self-polar if, for all $i = 1, \ldots, 5$, choosing p_i as a pole, the points p_{i+2} and p_{i+3} both lie on the equator. C. F. Gauss studied the geometry of such pentagons in a posthumously published work *Pentagramma Mirificum*.

2). The formulations of Theorems 1-4 are similar: if P is inscribed, or circumscribed, then $T_w(P)$ is projectively equivalent to P (or is circumscribed). Here w is a word in symbols $1, 2, 3, 4$, that varies from statement to statement, but in each case, w is *palindromic*: it is the same whether we read it left to right or right to left. This implies that, in each case, the transformation T_w is an involution: $T_w \circ T_w = Id$.

3). The statement of Theorem 1 can be rephrased as follows: *if P is an inscribed heptagon then $T_2(P)$ and $T_{12}(P)$ are projectively equivalent.* That is, the heptagon $Q = T_2(P)$ is equivalent to its projective dual Q^*. In fact, every projectively self-dual heptagon is obtained this way.

Similarly, Theorem 2 states: *if P is a circumscribed nonagon then $T_3(P)$ and $T_{13}(P)$ are projectively equivalent*, and hence $Q = T_3(P)$ is projectively self-dual. Once again, every projectively self-dual nonagon is obtained this way.

For odd n, the space of projectively self-dual n-gons in the projective plane, considered up to projective equivalence, is $n-3$-dimensional, see [1] (compare with $2n - 8$, the dimension of projective equivalence classes of all n-gons). The space of inscribed (or circumscribed) n-gons, considered up to projective equivalence of the conic, is also $n-3$-dimensional. Thus, for $n = 7$ and $n = 9$, we have explicit bijections between these spaces.

4). One may cyclically relabel the vertices of a polygon to deduce apparently new configuration theorems from Theorems 1-4. Let us illustrate this by an example. Rephrase the last statement of Theorem 4 as follows: *If P is an inscribed dodecagon then $T_{131313}(P)$ is also inscribed.* Now relabel the vertices as follows: $\sigma(i) = 5i \bmod 12$ (note that σ is an involution). The map T_3 is conjugated by σ as follows:

$$i \mapsto 5i \mapsto 5i + 3 \mapsto 5(5i + 3) = i + 3 \mod 12,$$

that is, the map is T_3 again, and the map T_1 becomes

$$i \mapsto 5i \mapsto 5i + 1 \mapsto 5(5i + 1) = i + 5 \mod 12,$$

that is, the map is T_5. We arrive at the statement: *If P is an inscribed dodecagon then $T_{535353}(P)$ is also inscribed.* Our java applet, cited above, shows pictures of this.

5). Theorem 4 appears to be a relative of a theorem in [4]: *Let P be a $4n$-gon whose odd sides pass through one fixed point and whose even sides pass through another fixed point. Then the $(2n-2)$nd iterate of the pentagram map T_{12} transforms P to a polygon whose odd vertices lie on one fixed line and whose even vertices lie on another fixed line.* Note that a pair of lines is a degenerate conic section. Note also that the dual polygon, $Q = T_1(P)$ is also inscribed into a pair of lines. Thus we have an equivalent formulation: *If Q is a $4n$-gon inscribed into a degenerate conic then $(T_1T_2)^{2n-2}T_1(Q)$ is also inscribed into a degenerate conic.*

We wonder if this results is a degenerate case of a more general theorem, much in the same way that Pappus's theorem is a degenerate case of Pascal's theorem.

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