Paper Moebius Bands with T Patterns

Richard Evan Schwartz *

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Abstract

This paper gives another proof of the key lemma in my recent paper which solves the optimal paper Moebius band conjecture of Halpern and Weaver, namely Lemma C

1 Introduction

The original content of this paper is completely superceded by my paper [**S1**], which solves the Halpern-Weaver conjecture about optimal paper Moebius bands. The paper [**S1**] has a discussion of this problem and many references.

The key lemma in [**S1**] is Lemma C below, a statement about the existence of so-called T-patterns in certain paper Moebius bands. A version of this lemma also appears in [**S2**]. The current version of this paper re-proves Lemma C in a different way than is done in [**S1**]. The proof we give here is similar to the one given in [**S2**] but now that I realize it the key result for the conjecture, I am taking the opportunity to re-do the proof more carefully and with better exposition. The proof here is longer than the proof given in [**S1**] but somehow more geometrically intuitive. Some readers interested in the proof of the Halpern-Weaver conjecture might like the argument here better.

An embedded *paper Moebius band* of aspect ratio λ is a smooth isometric embedding $I : M_\lambda \to \boldsymbol{R}^3$, where M_λ is the flat Mobius band

$$M_\lambda = ([0,1] \times [0,\lambda])/\sim, \qquad (x,0) \sim (1-x,\lambda) \tag{1}$$

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An *isometric mapping* is a map whose differential is an isometry. The map is an *embedding* if it is injective. In [**S1**] I prove that a smooth embedded paper Moebius band has aspect ratio greater than $\sqrt{3}$. This result is sharp, and the solution of the Halpern-Weaver conjecture.

Being a ruled surface, a smooth paper Moebius band has a foliation by straight line segments which we call *bends*. We call a bend *gentle* if the line extending it makes an angle of less than $\pi/4$ with the XY-plane. We call the Moebius band *gentle* if all the bends are gentle.

We say that a *T-pattern* in a paper Moebius band is a collection of 2 co-planar bends such that the lines extending them are perpendicular. Here is the key technical result of [**S1**].

Lemma 1.1 (C) *A gentle paper Moebius band has a T-pattern.*

In this paper I will reprove Lemma C in a different category, that of immersed piecewise linear Moebius bands. At the end, I will deduce Lemma C above from the polygonal version.

2 Nice Approximations

2.1 Pairs of Lines

In this first section we will prove two results about configuration spaces of lines. The second of these results, Lemma 2.2, will be useful in subsequent sections. After proving Lemma 2.2, we will define polygonal Moebius bands and then use Lemma 2.2 to show that we can approximate an arbitrary polygonal Moebius band by one which has a nice property useful for proving Lemma C.

Say that an *anchored line* in $\boldsymbol{R}^3$ is a line through the origin. Say that an *anchored plane* is a plane in $\boldsymbol{R}^3$ through the origin. Let Π_1 and Π_2 be anchored planes. A *sector* of the plane Π_j is a set linearly equivalent to the union of the $(++)$ and $(--)$ quadrants in $\boldsymbol{R}^2$. Let $\Sigma_j \subset \Pi_j$ be a sector. The boundary $\partial\Sigma_j$ is a union of two anchored lines crossing at the origin.

Lemma 2.1 *Suppose that Π_1 and Π_2 are not perpendicular. The set of perpendicular anchored lines (L_1, L_2) with $L_j \in \Pi_j$ for $j = 1, 2$ is diffeomorphic to a circle.*

Proof: For each anchored line $L_1 \in \Pi_1$ the line $L_2 = L_1^{\perp} \cap \Pi_2$ is the unique choice anchored line in Π_2 which is perpendicular to L_1. The line L_2 is a smooth function of L_1. So, the map $(L_1, L_2) \to L_1$ gives a diffeomorphism between the space of interest to us and a circle. ♠

Lemma 2.2 *Suppose Π_1 and Π_2 are not perpendicular and no line of $\partial\Sigma_1$ is perpendicular to a line of $\partial\Sigma_2$. Then the set of perpendicular pairs of anchored lines (L_1, L_2) with $L_j \in \Sigma_j$ for $j = 1, 2$ is either empty or diffeomorphic to a closed line segment. If (L_1, L_2) corresponds to an endpoint then exactly one of these lines lies in the boundary of its sector.*

Proof: Let Π_1 and Π_2 be anchored planes. Let S^1 denote the set of perpendicular pairs as in Lemma 2.1. Let $X \subset S^1$ denote the set of those pairs with $L_j \in \Sigma_j$. Let π_1 and π_2 be the two diffeomorphisms from Lemma 2.1. The set of anchored lines in Σ_j is a line segment and hence so is its inverse image $X_j \subset S^1$ under π_j. We have $X = X_1 \cap X_2$.

Suppose X is nonempty. Then some $p \in X$ corresponds to a pair of lines (L_1, L_2) with at most one $L_j \in \partial\Sigma_j$. But then we can perturb p slightly, in at least one direction, so that the corresponding pair of lines remains in $\Sigma_1 \times \Sigma_2$. This shows that $X_1 \cap X_2$, if nonempty, contains more than one point. But then the only possibility, given that both X_1 and X_2 are segments, is that their intersection is also a segment.

If neither L_1 nor L_2 lies in the boundary of its sector then we can perturb in both directions. This implies that X contains the point corresponding to (L_1, L_2) in its (relative) interior. Hence the endpoints of X correspond to pairs with at least one line in the boundary of a sector. Both lines cannot be in the sector boundary because, by assumption, a sector boundary line of one sector cannot be perpendicular to a sector boundary line of the other sector. ♠

2.2 Polygonal Moebius Bands

We represent M_λ as the quotient of a bilaterally symmetric trapezoid τ, as shown in Figure 2.1. A *transverse triangle* in M_λ is one having 1 edge ∂M_λ and 2 edges with their vertices in ∂M_λ as shown in Figure 2.1. We call the edge in ∂M_λ the *ridge*. We define a *pre-bend* of a transverse triangle to be a segment joining the ridge to the opposite vertex.

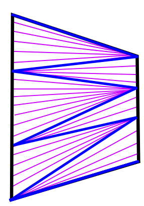

Figure 2.1: Transverse triangulation and pre-bend foliation

A *transverse triangulation* of M_λ is a partition of τ into transverse triangles. Each transverse triangle has a foliation by pre-bends, and these piece together to give the *pre-bend foliation* of τ. Say that *polygonal Moebius band* is a continuous map $I : M_\lambda \to \boldsymbol{R}^3$ that is piecewise affine with respect to some transverse triangulation of M_λ. We always cut open along a pre-bend to get the kind of trapezoid representation shown in Figure 2.1. The map I should be injective on each transverse triangle but not necessarily globally injective.

We define the *bends* to the images of the pre-bends under I. As in the smooth case, a *T-pattern* in a polygonal Moebius bend is a pair of bends having perpendicular and coplanar extending lines. More generally, we call two bends *partners* if the lines through the origin parallel to these bends are perpendicular.

We identify the pre-bends of M_λ with the circle $\boldsymbol{R}/\lambda\boldsymbol{Z}$ as follows: We map each pre-bend to its intersection with the centerline of M_λ, and this is a copy of $\boldsymbol{R}/\Lambda_Z$. We call two points $r, s \in \boldsymbol{R}/\lambda\boldsymbol{Z}$ *partners* if the bends $I(\beta_r)$ and $I(\beta_s)$ are partners. We let $\Omega \subset (\boldsymbol{R}/\lambda\boldsymbol{Z})^2$ be the subset of partner points. We call W *nice* if Ω is a piecewise smooth 1-manifold – i.e. a finite disjoint union of piecewise smooth embedded loops.

Lemma 2.3 *Let M be a polygonal Moebius band. We can find a linear transformation ϕ as close as we like to the identity so that $\phi(M)$ is nice.*

Proof: Say that an *image triangle* of M is the image under I of one of the triangles in the transverse triangulation. Each image triangle μ defines a sector. The anchored plane containing the sector is parallel to the one containing μ. The boundary of the sector is the union of the two anchored lines parallel to the apex-incident edges of μ.

Now we consider an affine adjustment using a linear map ϕ. Since we just need to destroy finitely many perpendicularity relations we can take ϕ as close as we like to the identity such that every pair of sectors associated to $\phi \circ I$ satisfies the hypotheses of Lemma 2.2. We also call the new polygonal Moebius band M and we show that it is nice.

Let Ω be the partner set. The space $(\boldsymbol{R}/\lambda\boldsymbol{Z})^2$ is tiled by *special rectangles* corresponding to pairs of transverse triangles. By Lemma 2.2 any nontrivial intersection of Ω with a special rectangle is a *special segment* with endpoints in the relative interiors of edges of ∂R. Any two special segments have disjoint interiors because their interiors lie in different rectangle interiors. Let s_1 be any special segment, contained in a special rectangle R_1. Let v be an endpoint of s_1. Let R_2 be the special rectangle adjacent to R_1 and sharing the edge containing v. Since $\Omega \cap R_2$ is nonempty, this intersection is another special segment s_2 which also contains v. In this manner, s_1 continues across v to a unique special segment s_2.

These properties, disjoint interiors and continuance across vertices, show that Ω is an embedded piecewise smooth 1-manfold. ♠

3 Existence of T Patterns

3.1 An Odd Homology Class

Let $I : M_\lambda \to M$ be a nice polygonal Moebius band. Let Ω be the partner set for M. By hypothesis, Ω is a piecewise smooth 1-manifold, a subset of the open cylinder Υ, which we get by removing the diagonal from $(\boldsymbol{R}/\lambda\boldsymbol{Z})^2$. We call Ω *odd* of Ω represents the nontrivial element of the homology group $H_1(\Upsilon; \boldsymbol{Z}/2) = \boldsymbol{Z}/2$. In this section we prove that Ω is odd.

We let $\overline{\Upsilon}$ be the compactification of Υ obtained by adding 2 boundary components. The point (a, b) lies near one boundary component if b lies just ahead of a in the cyclic order coming from $\boldsymbol{R}/\lambda\boldsymbol{Z}$. The point (a, b) lies near the other boundary component if b lies just behind of a in the cyclic order coming from $\boldsymbol{R}/\lambda\boldsymbol{Z}$. We get a path γ which runs from one boundary component of $\overline{\Upsilon}$ to the other by holding a fixed and varying b all the way around from ahead of a to just behind a. Let γ be such a path. If we pick a generically then γ intersects Ω transversely. In particular, γ intersects Ω a finite number of times.

Lemma 3.1 *If γ intersects Ω an odd number of times then Ω is odd.*

Proof: If ω is a component of Ω then γ intersects ω an even number or an odd number of times, depending respectively on whether ω is trivial or nontrivial in $H_1(\Upsilon; \boldsymbol{Z}/2)$. Hence Ω has an odd number of homologically nontrivial components. Hence Ω is odd. ♠

Lemma 3.2 *γ intersects Ω an odd number of times.*

Proof: We give an orientation to the pre-bend β_a corresponding to a. This gives an orientation to the bend $I(\beta_a)$. We attempt to give a continuous orientation to the bends $I(\beta_b)$, knowing that this is impossible because we are on a Moebius band. But we can almost do this. When b is just ahead of a we orient $I(\beta_b)$ so that it points almost in the same direction as $I(\beta_a)$. After we have gone all the way along γ until b is just behind a, the bend $I(\beta_b)$ points almost in the opposite direction as $I(\beta_a)$. This means that the bends are partners an odd number of times along the path. Hence γ intersects Ω an odd number of times. ♠

3.2 The Main Argument

Let M be the nice and gentle polygonal Moebius band. Suppose u, v are are partner bends. There is a unique pair of parallel planes U, V such that $u \subset U$ and $v \subset V$. These planes are both orthogonal to the cross product of vectors parallel to u and v. We call U or V *auxiliary planes*. If $U = V$ then we have a T-pattern.

Lemma 3.3 *No associated plane contains a vertical line.*

Proof: We argue by contradiction. Let (U, V) be a pair of associated planes which supposedly contain the vertical direction. Let (u, v) be the corresponding partner bends. Let L_u and L_v be the lines parallel to u and v through the origin. Let L^* be the line through the origin perpendicular to both L_u and L_v. Since L^* is perpendicular to all vectors in U and V, we know that L^* is perpendicular to $(0, 0, 1)$. Hence L^* lies in the XY-plane. We might as well rotate about the Z-axis so that L^* is the Y-axis. But then L_u and L_v lie in the XZ plane. So we have 2 lines in the XZ plane which are perpendicular and both make an angle of less than $\pi/4$ with the XY plane. This is impossible. ♠

Let us deduce the M has a T-pattern from these results. Consider $f : \Upsilon \to \Upsilon$ given by $f(a, b) = (b, a)$. This map is an involution and an isomorphism on $H_1(\Upsilon; \boldsymbol{Z}/2)$. By construction f permutes the components of Ω. Since Ω is odd, there must be some component ω of Ω such that $f(\omega) = \omega$. This means that we can find a continuous path in Ω, namely a suitable arc of ω, such that f swaps the endpoints of our path. Call this the *swapping path*.

Let (u_t, v_t) be the continuous path of pairs of partner bends in M corresponding to the swapping path. Let (U_t, V_t) be the corresponding continuous path of auxiliary planes. Let $[U_t]$ and $[V_t]$ be the points where these planes intersect the Z-axis. These points are well-defined and vary continuously by Lemma C2 and compactness. By construction these intersection points switch places as we traverse the swapping path. Hence there is some parameter s for which $[U_s] = [V_s]$. But then, because U_s and V_s are parallel, $U_s = V_s$. But then u_s and v_s make a T-pattern in M.

3.3 Approximation Arguments

We have just proved that a nice and gentle polygonal Moebius band has a T-pattern. Now let us get the same result without assuming niceness. Let M be a gentle polygonal Moebius band that is not necessarily nice. If we choose any linear ϕ close enough to the identity then $\phi(M)$ is also gentle. This is just compactness. By Lemma 2.3 we can choose such a ϕ so that $\phi(M)$ is both nice and gentle. So, we have a T-pattern on $\phi(M)$ by Lemma C1. The image of this T-pattern under ϕ^{-1} is as close as we like to being a T-pattern on M. Taking a limit we get a T-pattern on M. This completes the proof.

We have proved that a gentle polygonal Moebius band has a T-pattern. Let us now deduce Lemma C in the introduction from this fact. Suppose that $I : M_\lambda \to M$ is a smooth paper Moebius band that is gentle.

Take a finite list $\beta_1, ..., \beta_n$ of pre-bends in M_λ, with β_1 being the first bend and β_n being the last. Call this a *mesh*. These pre-bends divide M_λ into thin trapezoids. We add diagonals to get a transverse triangulation. We use the values of I on the vertices of the transverse triangles to define I'. By construction I and I' agree on $\beta_1, ..., \beta_n$. We call these bends the *shared bends*. Hence $I' : M_\lambda \to M'$ is a polygonal Moebius band.

If we take our mesh fine enough then by compactness and continuity M' will be gentle. But then M' has a T-pattern (u', v'). Once we take our mesh fine enough we can make u' and v' as close metrically as we like to shared bends u and v. Now we can say that M has a pair of bends whose extending lines are as close as we like to being perpendicular and coplanar. We can take a limit of these near T-patterns on M to get an actual T-pattern on M. This completes the proof.

4 References

[**S1**] R. E. Schwartz, *The Optimal Paper Moebius Band*, preprint, 2023

[**S1**] R. E. Schwartz, *An Improved Bound on the Optimal Paper Moebius Band*, Geometriae Dedicata, 2021.