On Nearly Optimal Paper Moebius Bands

Richard Evan Schwartz *

November 22, 2024

Abstract

Let $\epsilon < 1/384$ and let Ω be a smooth embedded paper Moebius band of aspect ratio less than $\sqrt{3}+\epsilon$. We prove that Ω is within Hausdorff distance $18\sqrt{\epsilon}$ of an equilateral triangle of perimeter $2\sqrt{3}$. This is an effective and fairly sharp version of our recent theorems in [**S0**] about the optimal paper Moebius band.

1 Introduction

A *smooth embedded paper Moebius band* is an infinitely differentiable isometric embedding $I : M_\lambda \to \boldsymbol{R}^3$, where M_λ is the flat Mobius band obtained by identifying the length-1 sides of a $1 \times \lambda$ rectangle. We set $\Omega = I(M_\lambda)$. The number λ is the *aspect ratio* of Ω. This terminology is a mouthful, so I will use the shorter term *paper Moebius band* to mean a smooth embedded paper Moebius band.

In [**S0**] I proved that a paper Moebius band has aspect ratio greater than $\sqrt{3}$. This bound was conjectured in 1977 by Halpern and Weaver in [**HW**]. The article [**FT**, §14] discusses the history of this conjecture and gives context for it. [**S0**] also has a discussion with many references.

I also proved a limiting result in [**S0**]: Let $\{I_n : M_{\lambda_n} \to \Omega_n\}$ be a sequence paper Moebius bands such that $\lambda_n \to \sqrt{3}$. Up to isometry, I_n converges uniformly to the triangular Moebius band map shown in Figure 1.

*Supported by N.S.F. Grant DMS-2102802 and also a Mercator Fellowship.

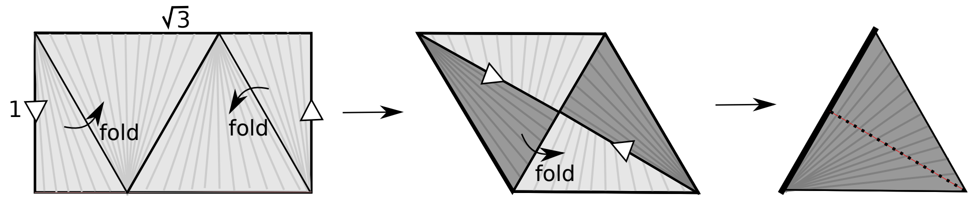

Figure 1: The triangular Moebius band.

The purpose of this article is to make the results from [**S0**] more effective. Let $I_0 : M_{\sqrt{3}} \to \Omega_{\sqrt{3}}$ be the triangular Moebius band. Given some other map $f : M_{\sqrt{3}} \to \boldsymbol{R}^3$ we let

$$\|I_0 - f\|_\infty = \sup_{p \in \partial M_{\sqrt{3}}} \|I_0(p) - f(p)\|. \tag{1}$$

Note that in Equation 1 we are only taking the sup over the boundary.

Theorem 1.1 *If $\epsilon < 1/4$ and $I : M_\lambda \to \Omega$ is a paper Moebius band with $\lambda < \sqrt{3} + \epsilon$ then there is a homeomorphism $\phi : \partial M_{\sqrt{3}} \to \partial M_\lambda$ and an isometry $\psi : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ such that $\|I_0 - \psi \circ I \circ \phi\|_\infty < 6\sqrt{\epsilon}$.*

Theorem 1.1 only deals with the boundary of Ω. Here is another result, with somewhat more restrictive conditions, that deals with the whole space.

Theorem 1.2 *Let $I : M_\lambda \to \Omega$ be a paper Moebius band with $\lambda < \sqrt{3} + \epsilon$. If $\epsilon < 1/4$ then there is an equilateral triangle $\bigtriangledown_0$ of perimeter $2\sqrt{3}$ such that every point of Ω is within $6\sqrt{\epsilon}$ of $\bigtriangledown_0$. If $\epsilon < 1/384$, then every point of $\bigtriangledown_0$ is within $18\sqrt{\epsilon}$ of Ω.*

Let me phrase Theorem 1.2 more gracefully. The *Hausdorff distance* between two compact subsets of $\boldsymbol{R}^n$ is defined to be the minimal ϵ such that each set is contained in the ϵ-tubular neighborhood of the other. (The minimum exists by compactness.) The Hausdorff distance gives a well-known metric on the set of compact subsets of $\boldsymbol{R}^n$. We are working with $n = 3$ in this paper.

Corollary 1.3 (Main) *Suppose $\epsilon < 1/384$ and $I : M_\lambda \to \Omega$ is a paper Moebius band with $\lambda < \sqrt{3} + \epsilon$. Then, in the Hausdorff metric, Ω is within $18\sqrt{\epsilon}$ of an equilateral triangle of perimeter $2\sqrt{3}$.*

Remarks: (1) In Theorem 1.1, I might have tried to get control over the entire map and not just the boundary map. I think it would be possible to do, with more effort and perhaps a worse estimate, but I wanted to keep the proof light. (2) The cutoff of 1/4 is somewhat arbitrary, but it seems like a reasonable notion of "fairly small". The significance of $\epsilon < 1/384$ is that then $6\sqrt{\epsilon} < 1/3$, the in-radius of the triangle $\bigtriangledown_0$. The fact that $384 = 18^2$ is not very significant; it is an artifact of the proofs. (3) I don't know how sharp the constants in Theorems 1.1 and 1.2 are but the $O(\sqrt{\epsilon})$ term is sharp. This derives from the fact that a right triangle with base 1 and hypotenuse $1+\epsilon$ has height $O(\sqrt{\epsilon})$. See §3.5 for more about this.

In §2 I will prove a few easy technical lemmas that will help with the overall proof, and then in §3 I will prove Theorems 1.1 and 1.2. To make this paper more self-contained, I recall the main arguments in [**S0**] in some detail. In §3.5 I give an example illustrating the sharpness of our results.

In [**S0**] I thanked many people for their help and insights into paper Moebius bands. I thank all these people again. This research is supported by a Simons Sabbatical Fellowship, a grant from the National Science Foundation, and a Mercator Fellowship. I'd like to thank all these organizations.

2 Some Perturbation Results

We let $\ell(\cdot)$ denote arc length, and we assume that $0 < \epsilon < 1/4$.

2.1 Two Trivialities

We will need two very easy estimates about the square root function. We single these out in advance to make the rest of our exposition go more smoothly.

Lemma 2.1 *If $L < 3/2$ then $\sqrt{L^2 + (13/4)\epsilon} > L + \epsilon$.*

Proof: $(L+\epsilon)^2 = L^2 + (2L+\epsilon)\epsilon < L^2 + (13/4)\epsilon$. ♠

Lemma 2.2 *If $L < 3/\sqrt{2}$ then $\sqrt{L^2 + (9/2)\epsilon} > L + \epsilon/2$.*

Proof: $(L + (\epsilon/2))^2 = L^2 + (L + (\epsilon/4))\epsilon < L^2 + (9/2)\epsilon$. ♠

2.2 Perturbing an Isosceles Triangle

Let $\bigtriangledown$ be a triangle with a horizontal base. We say that the *bottom vertex* of $\bigtriangledown$ is the vertex not on the base. Let $\bigtriangledown_*$ denote the isosceles triangle having the same base and height as $\bigtriangledown$. We get $\bigtriangledown_*$ from $\bigtriangledown$ by moving the bottom vertex horizontally by a distance we call $d(\bigtriangledown, \bigtriangledown_*)$. Let $\vee$ denote the union of the non-horizontal edges of $\bigtriangledown$. Likewise define $\vee_*$.

Lemma 2.3 *Suppose* $\ell(\vee_*) < 3$ *and the slopes of the sides of* $\vee_*$ *exceed* 1 *in absolute value. If* $d(\bigtriangledown, \bigtriangledown_*) \geq \sqrt{13\epsilon/2}$ *then* $\ell(\vee) > \ell(\vee_*) + 2\epsilon$.

Proof: Let p_1, p_2, q be the vertices of $\bigtriangledown$, with q being the bottom vertex. Likewise let q^* be the bottom vertex of $\bigtriangledown_*$. Let r_2 denote the reflection of p_2 in the horizontal line through q and q^*. By symmetry, $\ell(\vee_*) = \ell(\overline{p_1r_2})$ and $\ell(\vee) = \ell(\overline{p_2q}) + \ell(\overline{qr_2})$. By assumption, $\ell(\overline{p_1r_2}) < 3$ and the horizontal distance from q to $\overline{p_1r_2}$ is at least $\sqrt{13\epsilon/2}$. Since $\overline{p_1r_2}$ has slope at least 1 in absolute value, the distance from q to $\overline{p_1r_2}$ is at least $d = \sqrt{13\epsilon/4}$.

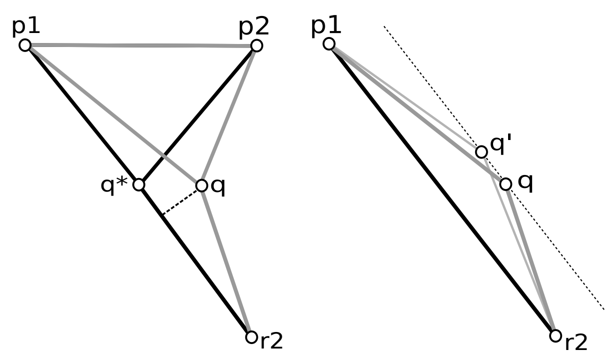

Figure 2: The quantities used in the proof.

Let $L = (1/2)\ell(\vee_*) < 3/2$. We form a new triangle T by sliding q parallel to $\overline{p_1r_2}$, to a new point q' which minimizes $\ell(\overline{p_1q'}) + \ell(\overline{q'r_2})$. This happens when the triangle (p_1, r_2, q') is isosceles. We have

$$\ell(\overline{p_1q'}) = \ell(\overline{q'r_2}) \geq \sqrt{L^2 + d^2} = \sqrt{L^2 + (13/4)\epsilon} > L + \epsilon.$$

The last inequality is Lemma 2.1. Hence

$$\ell(\vee) = \ell(\overline{p_1q}) + \ell(\overline{qr_2}) \geq \ell(\overline{p_1q'}) + \ell(\overline{q'r_2}) > 2L + 2\epsilon = \ell(\vee_*) + 2\epsilon.$$

This completes the proof. ♠

Remark: We call the trick of replacing q by q' the *sliding trick*. We will use this trick in the next section as well.

2.3 Perturbing an Affine Map

Let S be a line segment with $\ell(S) < 3$.

Let $I : S \to \boldsymbol{R}^3$ be a unit speed map. Let $I_* : S \to \boldsymbol{R}^3$ be the affine map that agrees with I on the endpoints. Let $S' = I(S)$ and $S'_* = I_*(S)$. Note that $\ell(S'_*) \leq \ell(S') = \ell(S) < 3$.

Consider the graph Γ of I defined as

$$\Gamma = \bigcup_{x \in S} (x, I(x)) \subset \boldsymbol{R}^4 = \boldsymbol{R} \times \boldsymbol{R}^3.$$

Likewise define Γ_*. The graph Γ_* is a straight line segment.

Lemma 2.4 *$\ell(\Gamma_*) \leq \ell(\Gamma) \leq 3\sqrt{2}$ and $\ell(\Gamma) - \ell(\Gamma_*) \leq \ell(S') - \ell(S'_*)$.*

Proof: The maps I and I_* respectively have speeds 1 and $s = \ell(S'_*)/\ell(S')$. As curves, Γ_* and Γ respectively have speeds $\sqrt{1+s^2}$ and $\sqrt{2}$. Integrating these functions over the domain, which has length $\ell(S')$, we see that

$$\ell(\Gamma_*) = \ell(S')\sqrt{1+s^2} = \sqrt{\ell(S')^2 + \ell(S'_*)^2} < 3\sqrt{2}. \tag{2}$$

$$\ell(\Gamma) = \ell(S')\sqrt{2} = \sqrt{\ell(S')^2 + \ell(S')^2} < 3\sqrt{2}. \tag{3}$$

That takes care of the first statement.

For the second statement, let $a = \ell(S')^2$ and let $f(t) = \sqrt{a+t^2}$. We have $|f'(t)| \leq 1$ for all t, regardless of the value of $a \geq 0$. This means that for $0 \leq t_* < t$ we have $f(t) - f(t_*) \leq t - t_*$. Applying this to $t = \ell(S')$ and $t_* = \ell(S'_*)$, we get $\ell(\Gamma) - \ell(\Gamma_*) \leq \ell(S') - \ell(S'_*)$. ♠

Let $\|\cdot\|_\infty$ denote the sup-norm on S.

Lemma 2.5 *If $\|I - I_*\|_\infty \geq 3\sqrt{\epsilon}$ then $\ell(S') > \ell(S'_*) + \epsilon$.*

Proof: By hypothesis, there is some $p \in S$ such that $\|I_*(p) - I(p)\| \geq 3\sqrt{\epsilon}$. Hence, there is a point $q \in \Gamma$ which is at least $3\sqrt{\epsilon}$ away from the point in Γ_* having the same $\boldsymbol{R}$ coordinate. Since I_* has speed $s < 1$, the graph Γ_* has "slope" at most 1. This implies that q is at least $\sqrt{9\epsilon/2}$ from Γ_* in $\boldsymbol{R}^4$.

Let $L = \ell(\Gamma_*)/2 < 3/\sqrt{2}$. By the same kind of point-sliding trick we used in the proof of Lemma 2.3, we have

$$\ell(S') - \ell(S'_*) \geq \ell(\Gamma) - \ell(\Gamma_*) \geq 2\sqrt{L^2 + (9/2)\epsilon} - 2L \geq \epsilon. \tag{4}$$

The first inequality if Lemma 2.4. The last inequality is Lemma 2.2. ♠

3 Proofs of the Results

3.1 The Optimality Proof Revisited

I first will recall some of the material in [**S0**] in order to give more self-contained proof of Theorems 1.1 and 1.2. Let $I : M_\lambda \to \Omega$ be a (smooth embedded) paper Moebius band. A *bend* on Ω is a straight line segment in Ω which has its endpoints (and only its endpoints) in $\partial\Omega$.

Lemma 3.1 *Ω has a foliation by bends.*

Proof: (Sketch) This seems to be a folklore result. See [**CL**], [**HM**], [**Mas**] for arguments which immediately work in case Ω has an open dense set of points with nonzero mean curvature. See [**S0**, Prop. 2.1] for the general case and more precise references. The basic idea is that each point on Ω of nonzero curvature has a unique tangent direction where the differential of the Gauss map is trivial. These directions integrate up to give a foliation of the nonzero-mean-curvature subset of Ω by line segments. The complementary pieces are planar trapezoids and one can fill in this foliation in an obvious way. ♠

A (embedded) *T-pattern* is a pair of disjoint bends on the paper Moebius band which lie in perpendicular intersecting lines.

Lemma 3.2 *Ω has a T-pattern.*

Proof: (Sketch) This is [**S0**, Lemma T]. The space of bends in the bend foliation is homeomorphic to a circle. The space of pairs of unequal bends has a 2-point compactification which makes it homeomorphic to the sphere S^2. Using the dot product and the cross product we define 2 odd functions on S^2 which, when they simultaneosly vanish, detect a T-pattern. By the Borsuk-Ulam Theorem, both functions do simultaneously vanish, and this gives the T-pattern. ♠

[**S0**, Lemma G] uses the T-pattern to show that $\lambda > \sqrt{3}$. We essentially reproduce this argument here.

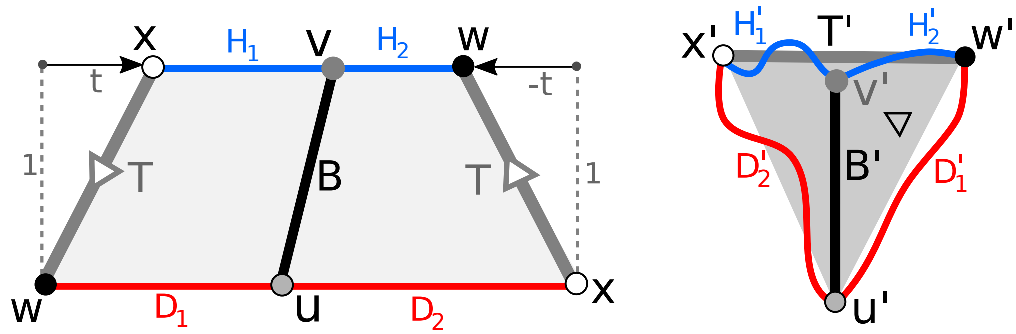

Figure 3: The trapezoid τ (left) and the T-pattern (right).

Let $S' = I(S)$ for any $S \subset M_\lambda$. We rotate Ω so that one of the bends of the T-pattern, T', lies in X-axis and the other bend, B', lies in the negative ray of the Y-axis. Let $B = I^{-1}(B')$ and $T = I^{-1}(T')$. These preimages are also line segments. We cut M_λ open along the line segment T to get a bilaterally symmetric trapezoid τ. We normalize τ so that the parallel sides are horizontal and so that u, v, w, x are mapped to Ω as in Figure 3. The quantity t is the horizontal displacement of T. Figure 3 shows the case when $t > 0$.

Note that the shaded triangle $\bigtriangledown$ on the right side of Figure 3 has base $\ell(T') = \sqrt{1+t^2}$ and height greater $\ell(B) \geq 1$. Let $\vee$ denote the union of the two non-horizontal sides of $\bigtriangledown$. The easy [**S0**, Lemma 2.2] says that $\ell(\vee) > \sqrt{5+t^2}$.

We have the following equations

$$\lambda = \ell(H) + t = \ell(D) - t,$$

$$\ell(H) = \ell(H') > \ell(T') = \ell(T) = \sqrt{1+t^2}$$
$$\ell(D) = \ell(D') \geq \ell(\vee) > \sqrt{5+t^2}. \tag{5}$$

By Equation 5,

$$\lambda > \max(h(t), d(t)), \qquad h(t) = \sqrt{1+t^2} + t, \qquad d(t) = \sqrt{5+t^2} - t. \tag{6}$$

Taking derivatives, we see easily that h is an increasing function and d is a decreasing function. Also $h(1/\sqrt{3}) = d(1/\sqrt{3}) = \sqrt{3}$. All this implies that $h(t) \geq \sqrt{3}$ if $t \geq 1/\sqrt{3}$ and $d(t) \geq \sqrt{3}$ if $t \leq 1/\sqrt{3}$. Hence $\lambda > \sqrt{3}$ regardless of the value of t. That proves Lemma G.

3.2 Geometric Bounds

Now we go beyond what we did in [**S0**]. If S is any object associated to our paper Moebius band Ω, let S_0 be the corresponding object associated to the triangular Moebius band Ω_0. In particular, let (B_0', T_0') be the T-pattern associated to Ω_0. These are normalized as above. We take ψ to be the translation so that $\psi(T')$ and T_0' have the same midpoint. (Both are segments in the X-axis.) Let $t_0 = 1/\sqrt{3}$.

Lemma 3.3 $|t - t_0| < 4\epsilon/3$. *In particular* $t \in (0,1)$.

Proof: The functions h and d are both convex, and $h'(t_0) = 3/2$ and $d'(t_0) = -3/4$. By convexity, $|h'(t)| \geq 3/2$ when $t \geq t_0$ and $|d'(t)| \geq 3/4$ when $t \leq t_0$. Under our assumption that $\lambda < \sqrt{3} + \epsilon$ we get $|t - t_0| < 4\epsilon/3 \leq 1/3$. The second bound now follows from the fact that $1/3 < t_0 < 2/3$. ♠

Lemma 3.4 (Length Bound) $\ell(H) < \ell(D) < 3$.

Proof: Since $t > 0$ we have $\ell(H) < \ell(D)$. Since $\epsilon \leq 1/4$ we have $\lambda < 2$. Since $t < 1$ have $\ell(D) = \lambda + t < 2 + 1 = 3$. ♠

Lemma 3.5 $|\ell(T') - 2t_0| < \epsilon$.

Proof: We have $\ell(T') = f(t) := \sqrt{1+t^2}$ and $|t - t_0| < 4\epsilon/3$. Also, we have $f(t_0) = 2t_0$. At the same time, $|f'(s)| < 3/4$ on $(0,1)$, an interval that contains t and t_0. Hence, Lemma 3.3 gives $|\ell(T') - 2t_0| < \epsilon$, as claimed. ♠

Lemma 3.6 *The height y of $\bigtriangledown$ is less than* $1 + \epsilon$.

Proof: Let $d(y,t) = \sqrt{1 + t^2 + 4y^2} - t$. For each choice of y the function $d(*, t)$ is decreasing. The obvious generalization of our proof of Lemma G above gives $\lambda > h(t_y)$ where t_y is such that $h(t_y) = d(y, t_y)$. We have $t_y = \pm y^2/\sqrt{1+2y^2}$. The negative choice leads to $\lambda > \sqrt{5} > \sqrt{3} + \epsilon$. The positive choice leads to $\lambda > g(y) := h(t_y) = \sqrt{1+2y^2}$. Now $g(1) = \sqrt{3}$ and $g'(y) > 1$ for $y \geq 1$. Since $\lambda < \sqrt{3} + \epsilon$ we have $y < 1 + \epsilon$. ♠

Let $\bigtriangledown_*$be as in Lemma 2.3. Let $\delta = d(\bigtriangledown, \bigtriangledown_*)$.

Lemma 3.7 $\delta < \sqrt{13\epsilon/2}$.

Proof: We use the notation from Lemma 2.3. Assume that $\delta \geq \sqrt{13\epsilon/2}$. Note that

$$\ell(\vee_*) \leq \ell(\vee) \leq \ell(D') = \ell(D) < 3.$$

The base of $\bigtriangledown$ is $\sqrt{1+t^2} < 2$, and the height of $\bigtriangledown$ is at least 1. Hence the absolute values of the slopes of the sides of $\vee_*$ are at least 1. Since $\delta \geq \sqrt{13\epsilon/2}$ we now have all the hypotheses of Lemma 2.3, which tells us that $\ell(\vee) - \ell(\vee_*) > 2\epsilon$. This also tells us that $\ell(\bigtriangledown) - \ell(\bigtriangledown_*) > 2\epsilon$.

If $t \geq t_0$ then $\ell(\bigtriangledown_*) \geq 2\sqrt{3}$, and so

$$2\lambda = \ell(H) + \ell(D) = \ell(H') + \ell(D') \geq \ell(\bigtriangledown) > \ell(\bigtriangledown_*) + 2\epsilon \geq 2\sqrt{3} + 2\epsilon.$$

If $t < t_0$ then we consider the function $d(t)$ from Equation 6 and observe that

$$\lambda \geq \ell(\vee) - t > \ell(\vee_*) - t + 2\epsilon > d(t) + 2\epsilon \geq \sqrt{3} + 2\epsilon.$$

Either case gives $\lambda > \sqrt{3} + \epsilon$, a contradiction. Hence $\delta < \sqrt{13\epsilon/2}$. ♠

Lemma 3.8 (T Pattern Bound) *Each endpoint of the T-pattern (T, B) is within $3\sqrt{\epsilon}$ of the corresponding endpoint of (T_0, B_0).*

Proof: By Lemma 3.5, we have $\|x'_0 - x'\| < \epsilon/2$ and $\|w'_0 - w'\| < \epsilon/2$. This takes care of the endpints of T'.

Since $\bigtriangledown$ has height less than $1 + \epsilon$, the y-coordinates of the endpoints of B' are within ϵ of the y-coordinates of the corresponding endpoints of B'_0. From Lemma 3.7, the x-coordinates of the endpoints of B' are within $\sqrt{13\epsilon/2}$ of the x-coordinates of B'_0. By the Pythagorean Theorem,

$$\|u' - u'_0\|, \|v' - v'_0\| < \sqrt{(13/2)\epsilon + \epsilon^2} < 3\sqrt{\epsilon}. \tag{7}$$

The final inequality is equivalent to the statement that $p(\epsilon) = (5/2)\epsilon - \epsilon^2$ is positive for $\epsilon \in (0, 1/4)$, and this is certainly true. ♠

3.3 Proof of Theorem 1.1

We have our paper Moebius band

$$I : M_\lambda \to \Omega.$$

We have quadrilaterals τ and τ_0 as in the left side of Figure 3. We think of $\partial\tau$ as having 6 edges: In addition to H_1, H_2, D_1, D_2 we have the non-horizontal edges which we call T_1 and T_2. By construction $T_1' = T_2' = T'$. We make the same labelings for τ_0. We define $I_* : \partial\tau \to \boldsymbol{R}^3$ to be the unique map that is affine on each edge of τ. With this definition, I_* induces a piecewise linear map on the union ∂M_λ.

Lemma 3.9 *$\|I_* - I\|_\infty < 3\sqrt{\epsilon}$.*

Proof: Here, as in Theorem 1.1, we are taking the sup-norm over points on ∂M_λ. For each $S \in \{H_1, H_2, D_1, D_2\}$ we have $\ell(S) < 3$. Since $\lambda < \sqrt{3} + \epsilon$ we must have $\ell(S') < \ell(S_*') + \epsilon$. This is to say that we cannot have more than ϵ of slack in any of these curves. Now we conclude, by Lemma 2.5, that

$$\sup_{p \in S} \|I_*(p)\| < 3\sqrt{\epsilon}.$$

Since this works for all choices of S, we get the bound of this lemma. ♠

We define $\phi : \partial M_{\sqrt{3}} \to \partial M_\lambda$ to be the piecewise linear map which maps each edge of τ_0 to the corresponding edge of τ. Lemma 3.9 gives us

$$\|I \circ \phi - I_* \circ \phi\|_\infty = \|I - I_*\|_\infty < 3\sqrt{\epsilon}. \tag{8}$$

Finally, let us compare $I^* \circ \phi$ to I_0. Both these maps are affine on each of the edges of τ_0. Since these maps differ by at most $3\sqrt{\epsilon}$ on the endpoints of each edge of τ, we have $\|I_0 - I_* \circ \phi\|_\infty < 3\sqrt{\epsilon}$. Combining this with Equation 8 we have

$$\|I_0 - I \circ \phi\|_\infty \leq \|I_0 - I_* \circ \phi\|_\infty + \|I_* \circ \phi - I \circ \phi\|_\infty < (3+3)\sqrt{\epsilon} < 6\sqrt{\epsilon}. \tag{9}$$

This completes the proof of Theorem 1.1,

3.4 Proof of Theorem 1.2

By Theorem 1.1, each of these bends foliating Ω has its endpoints inside the $6\sqrt{\epsilon}$-tubular neighborhood N of the triangle $\bigtriangledown_0$. But N is convex, and hence all the bends foliating Ω lie within N. Hence $\Omega \subset N$. This proves that every point of Ω is within $6\sqrt{\epsilon}$ of $\bigtriangledown_0$. This is the first statement of Theorem 1.2.

For the second statement, we take $\epsilon < 1/384$. Note that $d := 6\sqrt{\epsilon} < 1/3$, and $1/3$ is the in-radius of $\bigtriangledown_0$. Figure 4 shows a picture of an annular neighborhood A of $\bigtriangledown_0$. The core curve of A is $\bigtriangledown_0$. The disks, centered at the vertices of $\bigtriangledown_0$, each have radius d. The fact that $d < 1/3$ guarantees that Figure 4 is a topologically accurate picture. Let Dx' be the disk which is centered at x', etc. Let C be the solid triangle bounded by the inner boundary of A.

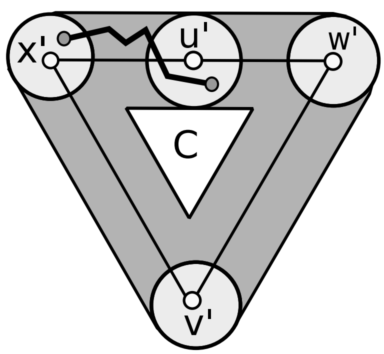

Figure 4: The neighborhood A of $\bigtriangledown_0$.

Let $f : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ denote projection. By Theorem 1.1, we see that $f(\partial\Omega) \subset A$. In particular, it makes to ask about which element in the fundamental group $\pi_1(\boldsymbol{R}^2 - C)$ this loop represents.

Lemma 3.10 *$f(\partial\Omega)$ generates $\pi_1(\boldsymbol{R}^2 - C)$.*

Proof: Consider the segment H_1 of $\partial\tau$. The image H_1' is a curve which connects a point in Dx' to a point in Du' and remains in the portion of A connecting these disks in the shortest way. See Figure 4. In other words, H_1' cannot "wind the other way" around C. The curves H_2', D_1', D_2' have similar properties. From this we see that $f(\partial\Omega)$ and $f(\partial\Omega_0)$ represent the same element in the fundamental group $\pi_1(\boldsymbol{R}^2 - C)$, namely a generator. ♠

Lemma 3.11 *$C \subset f(\Omega)$.*

Proof: Suppose not. Then there is $p \in C$ such that $f(\Omega) \subset \boldsymbol{R}^2 - p$. Note that $f(\partial\Omega)$ also generates $\pi_1(\boldsymbol{R}^2 - p)$. Let Λ be the core curve of Ω, the curve connecting all the midpoints of the bends. Note also that there is a homotopy from $f(\partial\Omega)$ to a curve which winds twice around $f(\Lambda)$. We just push in along the bends. Call this double-wrapped curve $f(2\Lambda)$. Since our homotopy avoids p, we see that $f(2\Lambda)$ generates $\pi_1(\boldsymbol{R}^2 - p) = \boldsymbol{Z}$. This is impossible because $f(2\Lambda)$ is an even number in $\pi_1(\boldsymbol{R}^2 - p)$. ♠

By the previous result, every vertical line through a point of C intersects Ω. But all of Ω lies in the $6\sqrt{\epsilon}$ neighborhood of $\boldsymbol{R}^2$. Hence every point of C is within $6\sqrt{\epsilon}$ of a point of Ω. But every point of ∇_0 is within $12\sqrt{\epsilon}$ of a point of C. Hence every point of ∇_0 is within $18\sqrt{\epsilon}$ of a point of Ω.

This completes the proof of Theorem 1.2.

3.5 Sharpness of the Results

Now we give an example illustrating the sharpness of our results. We will describe a polygonal paper Moebius band. Using the *pseudo-fold method* in [**HW**] (which just amounts to smoothing out the corners) we can then approximate this object as close as we like by smooth embedded paper Moebius bands. We omit the details of the smoothing.

We start with $M_{\sqrt{3}}$ and we insert a vertical strip B of width $O(\epsilon)$ as shown on the left in Figure 5.

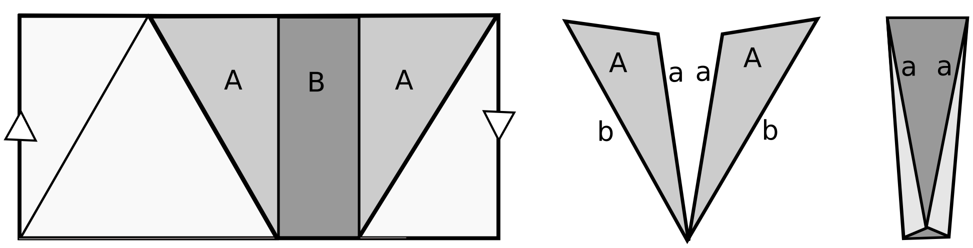

Figure 5: Modified triangular Moebius band.

We start with the triangle ∇_0 and slit it open along its vertical midline. This leaves us with two triangular "doors". Each door has a side which coincides with a side of $\vee_0$. Call these the b-sides. We rotate these doors about the b-sides by an angle $O(\sqrt{\epsilon})$. The two vertical sides come out of the plane and make a very sharp V. We call this V the *the crack*. The width of the crack is $O(\epsilon)$ and it rises $O(\sqrt{\epsilon})$ out of the plane. This derives from the fact that $1-\cos(\epsilon)=O(\epsilon^2)$ and $\sin(\epsilon)=O(\epsilon)$. We have labeled the sides of the crack by the letter a. We call the union of rotated triangles *the saloon doors*.

At the same time, we fold up B so that its two vertical edges become the sides of the same kind of V. We adjust the width, keeping it $O(\epsilon)$, so that the new V is isometric to the original one, the one we call the crack. We call this folded image of B (which is still planar) *the plug*. We now revolve the plug π radians about the Z-axis, tilt slightly, and glue it to the crack along the a-sides. This creates a piecewise isometric map from the ABA quadrilateral on the left of Figure 5 to a region whose two outer bends comprise $\vee_0$. Our image, which we call *the wrinkle*, rises $O(\sqrt{\epsilon})$ out of the plane.

Finally, we build the triangular Moebius band and replace the relevant copy of $\bigtriangledown_0$ by the wrinkle. For the purposes of smooth approximation, we take care that (in an infinitesimal sense) the wrinkle is "on the outside" so that nearby approximations will be embedded. By construction, our new Ω has aspect ratio $\lambda+O(\epsilon)$, contains $\bigtriangledown_0$, and also contains points $O(\sqrt{\epsilon})$ from $\bigtriangledown_0$. The smoothings of Ω are the examples showing the sharpness of $O(\sqrt{\epsilon})$ in our results.

4 References

[**CL**], S.-S. Chern and R. K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957) pp 306–318

[**FT**], D. Fuchs, S. Tabachnikov, *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*, AMS 2007

[**HN**], P. Hartman and L. Nirenberg, *On spherical maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959) pp 901–920

[**HW**], B. Halpern and C. Weaver, *Inverting a cylinder through isometric immersions and embeddings*, Trans. Am. Math. Soc **230**, pp 41–70 (1977)

[**Mas**] W. S. Massey, *Surfaces of Gaussian Curvature Zero in Euclidean* 3-*Space*, Tohoku Math J. (2) 14 (1), pp 73-79 (1962)

[**S0**] R. E. Schwartz, *The optimal paper Moebius band*, Annals of Math. (2024) to appear (see also arXiv 2308:12641)