Unbounded Orbits in the Plaid Model

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Abstract

This is a sequel to [**S0**]. In this paper I prove that the plaid model has unbounded orbits at all irrational parameters. This result is closely related to my result [**S1**] that outer billiards has unbounded orbits with respect to any irrational kite.

1 Introduction

I introduced the plaid model in [**S0**], and this paper is a sequel. The plaid model is closely related to outer billiards on kites [**S1**], somewhat related to DeBruijn's pentagrids [**DeB**], and also somewhat related to corner percolation and P. Hooper's Truchet tile system [**H**].

The plaid model has 2 descriptions. For a rational parameter $p/q \in (0,1)$ with pq even, there is one description in terms of intersection points on grids of lines. See §2. The output of the plaid model is a union of embedded polygons in the plane which we call *plaid polygons*.

At the same time, there is a 3 dimensional polyhedron exchange transformation $\widehat{X}_P$ which exists for all $P \in (0,1)$. This PET has the property that the plaid polygons associated to the parameter p/q correspond to the so-called vector dynamics of a distinguished set of orbits of $\widehat{X}_P$ for $P = 2p/(p+q)$. This is [**S0**, Theorem 1.1]. See §3.

We can interpret the vector dynamics at irrational parameters as (Hausdorff) limits of these plaid polygons. We call these limits *plaid paths*. In

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other words, the plaid paths are the vector dynamics for the PET at all parameters, and for even rational parameters they are embedded closed loops. All this is explained in §2-3.

The purpose of this paper is to prove

Theorem 1.1 (Unbounded Orbits) *For every irrational* $P \in (0,1)$ *the PET* $\widehat{X}_P$ *has an infinite orbit. The corresponding plaid path has infinite diameter projections in both coordinate directions.*

The reason one might want the coordinate projections of a plaid path to be unbounded is that the projection into the x-coordinate has a dynamical interpretation in terms of outer billiards orbits. According to the Quasi-Isomorphism Conjecture in [**S0**], a plaid path with unbounded x-projection corresponds to an unbounded orbit for outer billiards on the kite with the same parameter.

In addition to the Unbounded Orbits Theorem, we will prove a number of results about how the set of plaid polygons associated to a given parameter contains pieces of the plaid polygons associated to a different and closely related parameter. I mean *closely related* in the sense of Diophantine approximation. These copying results are difficult to state without a buildup of terminology; in the paper they are called the Copy Theorem, and the three Copy Lemmas.

The proof of the Unbounded Orbits Theorem here is similar in spirit to the proof in [**S1**] of the Unbounded Orbits Theorem for outer billiards on kites. One difference is that the proof here is traditional whereas the proof in [**S1**] was heavily computer-assisted. (The proof here quotes some results from [**S0**], but these results are also proved in a traditional way.) The plaid model is somehow more straightforward and transparent than the corresponding object from [**S1**], namely the *arithmetic graph*, and so it is possible to do all the calculations by hand.

On the other hand, the results here are like the results in [**S1**] in that we discovered them through experimentation, and checked practically every step on many examples before trying for proofs. As I remarked in [**S1**], the massive reliance on the computer for motivation is a virtue rather than a bug: I checked all the identities in the paper on thousands of examples.

The Quasi-Isomorphism Conjecture is a piece of unfinished business from [**S0**]. This result says that, for every parameter, there is a bijection between the set of plaid polygons and the set of components of the arithmetic graph which differs from the identity map by at most 2 units. I hope to prove the Quasi-Isomorphism Theorem next. Given the truth of the Quasi-Isomorphism Conjecture, the Unbounded Orbits Theorem here is very nearly equivalent to the one in [**S1**], but perhaps there are a few small details which would need to be worked out.

Here is an overview of the paper.

- In §2, we discuss the definition of the plaid model in terms of grids of lines. In particular, we give a new interpretation of the plaid model in terms of triple intersections of lines which are decorated with signs.

- In §3, we discuss the plaid model in terms of polyhedron exchange transformations. The material here is the same as in [**S0**] except that we discuss geometric limits in more detail.

- In §4 we prove the Unbounded Orbits Theorem modulo some elementary number theory, and and two auxiliary results, the Box Lemma and the Copy Theorem.

- In §5 we take care of the number-theoretic component of the proof. In particular, we prove Lemma 5.1, a multi-part result which summarizes a number of small number-theoretic identities and inequalities between rational numbers.

- In §6 we reduce the Box Lemma and the Copy Theorem to three similar but smaller results, which we call the Weak, Strong, and Core Copy Lemmas.

- In §7 we establish the Weak and Strong Copy Lemmas. This is an application of the signed crossing interpretation of the Plaid Model.

- In §8 we establish the Core Copy Lemma. This is another application of the signed crossing interpretation.

2 The Grid Description

2.1 Basic Definitions

Even Rational Parameters: We will work with $p/q \in (0,1)$ with pq even. We call such numbers *even rational parameters*. We will work closely with the auxiliary quantities

$$\omega = p + q, \qquad P = \frac{2p}{\omega}, \qquad Q = \frac{2q}{\omega}. \tag{1}$$

Four Families of Lines We consider 4 infinite families of lines.

- $\mathcal{H}$ consists of horizontal lines having integer y-coordinate.
- $\mathcal{V}$ consists of vertical lines having integer x-coordinate.
- $\mathcal{P}$ is the set of lines of slope $-P$ having integer y-intersept.
- $\mathcal{Q}$ is the set of lines of slope $-Q$ having integer y-intersept.

Adapted Functions: Let $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_1$ denote the sets of even and odd integers respectively. We define the following 4 functions:

- $F_H(x,y) = 2Py \bmod 2\boldsymbol{Z}$
- $F_V(x,y) = 2Px \bmod 2\boldsymbol{Z}$.
- $F_P(x,y) = Py + P^2x + 1 \bmod 2\boldsymbol{Z}$
- $F_Q(x,y) = Py + PQx + 1 \bmod 2\boldsymbol{Z}$.

We call F_H and F_V *capacity functions* and F_P and F_Q *mass functions*. We shall never be interested in the inverse images of $\boldsymbol{Z}_1$, so we will always normalize so that our functions take values in $(-1,1)$.

Intersection Points: Recall that $\omega = p + q$. Let

$$\Omega_j = \boldsymbol{Z}_j \omega^{-1}. \tag{2}$$

We observe that, for each index $A \in \{H, V\}$ and $B \in \{P, Q\}$, we have $F_A^{-1}(\Omega_0) = \mathcal{A}$ and $F_B^{-1}(\Omega_1) = \mathcal{B}$.

We call z an *intersection point* if z lies on both an A line and a B line. In this case, we call z *light* if

$$|F_B(z)| < |F_A(z)|, \qquad F_A(z)F_B(z) > 0. \tag{3}$$

Otherwise we call z *dark*. We say that B is the *type* of z. Thus z can be light or dark, and can have type P or type Q. When z lies on both a $\mathcal{P}$ line and a $\mathcal{Q}$ line, we say that z has both types.

Capacity and Mass: For any index $C \in \{V, H, P, Q\}$, we assign two invariants to a line L in $\mathcal{C}$, namely $|(p+q)F_C(L)|$ and the sign of $F_C(L)$. We call the second invariant the *sign* of L in all cases. When $A \in \{V, H\}$, we call the first invariant the *capacity* of L. When $A \in \{P, Q\}$ we call the first invariant the *mass* of L. The masses are all odd integers in $[1, p+q]$ and the capacities are all even integers in $[0, p+q-1]$. Thus, an intersection point is light if and only if the intersecting lines have the same sign, and the mass of the one line is less than the capacity of the other.

Double Counting Midpoints: We introduce the technical rule that we count a light point twice if it appears as the midpoint of a horizontal unit segment. The justification for this convention is that such a point is always a triple intersection of a $\mathcal{P}$ line, a $\mathcal{Q}$ line, and a $\mathcal{H}$ line; and moreover this point would be considered light either when computed either with respect to the $\mathcal{P}$ line or with respect to the $\mathcal{Q}$ line. See Lemma 5.2 below.

Good Segments: We call the edges of the unit squares *unit segments*. Such segments, of course, either lie on $\mathcal{H}$ lines or on $\mathcal{V}$ lines. We say that a unit segment is *good* if it contains exactly one light point. We say that a unit square is *coherent* if it contains either 0 or 2 good segments. We say that the plaid model is *coherent* at if all squares are coherent for all parameters. Here is the fundamental theorem concerning the plaid model. [**S0**, Fundamental Theorem] says that the plaid model is coherent at all parameters.

Plaid Polygons: In each unit square we draw the line segment which connects the center of the square with the centers of its good edges. The squares with no good edges simply remain empty. We call these polygons the *plaid polygons*.

2.2 Symmetries

The Symmetry Lattice We fix some even rational parameter p/q. Let $\omega = p + q$ as above. Let $L \subset \boldsymbol{Z}^2$ denote the lattice generated by the two vectors

$$(\omega^2, 0), \qquad (0, \omega). \tag{4}$$

We call L the *symmetry lattice*.

Blocks: We define the square $[0, \omega]^2$ to be the *first block*. The pictures above always show the first block. In general, we define a *block* to be a set of the form $B_0 + \ell$, where B_0 is the first block and $\ell \in L$. With this definition, the lattice L permutes the blocks. We define the *fundamental blocks* to be $B_0, ..., B_{\omega-1}$, where B_0 is the first block and

$$B_k = B_0 + (k\omega, 0). \tag{5}$$

The union of the fundamental blocks is a fundamental domain for the action of L. We call this union the *fundamental domain*.

Translation Symmetry: All our functions F are L-invariant. That is, $F(v + \ell) = F(v)$ for all $\ell \in L$. This follows immediately from the definitions. Note that a unit segment in the boundary of a block does not have any light points associated to it because $F_H = 0$ on the horizontal edges of the boundary and $F_V = 0$ on the vertical edges of the boundary. (A vertex of such a segment might be a light point associated to another unit segment incident to it but this doesn't bother us.) Hence, every plaid polygon is contained in a block, and is translation equivalent to one in a fundamental block.

Reflection Symmetry: Let $\widehat{L}$ denote the group of isometries generated by reflections in the horizontal and vertical midlines (i.e. bisectors) of the blocks. In §2 we proved that the set of plaid polygons is invariant under the action of $\widehat{L}$. Regarding the types of the intersection point (P or Q), the reflections in $\widehat{L}$ have the following action:

- They preserve the types of the intersection points associated to the horizontal lines.

- They interchange the types of the intersection points associated to the vertical lines.

2.3 An Alternate Description

We have defined the plaid model in terms of the $\mathcal{H}$ and $\mathcal{V}$ lines, and the $\mathcal{P}$ and $\mathcal{Q}$ lines. Our description favored lines of negative slope. To clarify the situation, we refer to the $\mathcal{P}$ and $\mathcal{Q}$ lines as the $\mathcal{P}_-$ and $\mathcal{Q}_-$ lines. We define the $\mathcal{P}_+$ and $\mathcal{Q}_+$ lines to be the lines having integer y-intercept and slope $+P$ and $+Q$ respectively

We define the mass of a $\mathcal{P}_+$ line L_+ so that it equals the mass of the $\mathcal{P}_-$ line L_- having the same y-intercept, and we define the sign of the L_+ to be opposite the sign of L_-. We make the same definitions for the $\mathcal{Q}_+$ lines.

Here is a result that is implicitly contained in [**S0**], but we want to make it explicit here.

Lemma 2.1 (Vertical) *Let z be an intersection point on a $\mathcal{V}$ line. Then either z is the intersection of a $\mathcal{P}_+$ line with a $\mathcal{Q}_-$ line or z is the intersection of a $\mathcal{P}_-$ line with a $\mathcal{Q}_+$ line. Moreover z is a light particle if and only if the three lines containing z have the same sign.*

Proof: We have the functions

$$F_{P,-}(x,y) = [Py + P^2x + 1]_2, \qquad F_{Q,-}(x,y) = [Py + PQx + 1]_2,$$

corresponding to the $\mathcal{P}_-$ and $\mathcal{Q}_-$ lines. There are similar functions corresponding to the $\mathcal{P}_+$ and $\mathcal{Q}_+$ lines, namely:

$$F_{P,+}(x,y) = [-Py + P^2x + 1]_2, \qquad F_{Q,+}(x,y) = [-Py + PQx + 1]_2,$$

Let's say that z lies on the intersection of a $\mathcal{V}$ line and a $\mathcal{P}_-$ line. Then $\omega F_{P,-}(z)$ is an odd integer in $(-\omega,\omega)$ and $\omega F_{V,-}(z)$ is an even integer in $(-\omega,\omega)$.

But then we have the identity

$$F_{P,-} + F_{Q,+} = [2Px]_2 = F_H. \tag{6}$$

This tells us that $\omega F_{Q,+}(z)$ is also an odd integer and so z lies on some $\mathcal{Q}_+$ line. If all three lines have the same sign, then Equation 6 forces $F_{P,-}(z) < F_H(z)$. So z is a light particle. If the $\mathcal{P}_-$ line and the $\mathcal{V}$ line have opposite signs then by definition z is dark. If the $\mathcal{Q}_+$ line and the $\mathcal{V}$ line have opposite signs, then Equation 6 forces $F_{P,-}(z) > F_H(z)$ and again z is dark. ♠

The Inert Principle: Equation 6 has one more consequence of importance to us. The $\mathcal{P}$ and $\mathcal{Q}$ lines through the points in $\omega\boldsymbol{Z}$ have mass ω, but it is impossible to give them a sign. We call these lines *inert* and we call the intersection points on them inert as well. All the inert intersection points are dark. Thanks to Equation 6, one of the following is true for each inert intersection point z.

- z lies in the boundary of a block, and the two slanting lines containing it are both inert.
- The other slanting line containing z has the opposite sign as the horizontal/vertical line containing z.

We call this dichotomy the *Inert Principle*.

The Vertical Lemma and the Inert Principle tell us that the types (light versus dark) of the vertical intersections in the plaid model are entirely determined by the signs of the grid lines. The same goes for the horizontal intersections, though the condition on the signs is different: The $\mathcal{H}/\mathcal{V}$ lines and the $(-)$ lines should have the same sign and the $(+)$ lines should have the opposite sign. Of course, we have set our sign conventions to favor a simpler description of the vertical intersections.

2.4 The Mass and Capacity Sequences

Now we will package the information in the last section in a way that will be more useful to us. Let

$$\Sigma = [x_0, x_1] \times [y_0, y_1], \tag{7}$$

We assume that Σ does not intersect the vertical boundary of a block. More precisely, for this paper we will have $1 = x_0 < x_1 < \omega$.

All the $\mathcal{V}$ lines which intersect Σ have positive capacity. The signs could be positive or negative. We define two sequences $\{c_j\}$ and $\{m_j\}$ w.r.t. Σ. We have the *capacity sequence*

$$c_j = [2Pj]_2, \qquad j = x_0, ..., x_1. \tag{8}$$

The terms of the capacity sequence all lie in $(-1, 1)$.

We also have the *mass sequence*

$$m_j = [P_j + 1]_2, \qquad j = y_0 - 2x + 1, ..., y_1 + 2x - 1, \qquad x = x_0 - x_1. \tag{9}$$

Remarks:
(i) Notice that the indices for the mass sequence start below the bottom edge of Σ (so to speak) and end above it. This is important for us for reasons which will become clear momentarily.
(ii) We will allow $j \in \omega \boldsymbol{Z}$ in the mass sequence. Such terms do not have a well-defined sign. The corresponding slanting lines are inert. When we speak of the signs of the terms of the mass sequence, we mean to ignore these terms.
(iii) Really we only care about the signs in the mass and capacity sequences, but for the purposes of running certain kinds of arguments it is useful to keep track of the numerical values as well.

Lemma 2.2 *The shade of any vertical intersection point in Σ is determined by the signs of the terms in the mass and capacity sequences.*

Proof: Since the slanting lines in the plaid model have slopes in $(-2, 2)$, every slanting line which contains a point of Σ intersects the y-axis in the interval

$$\{0\} \times [y_0 - 2x + 1, y_1 + 2x - 1].$$

Hence, every vertical intersection point lies on 3 lines whose signs are all determined by the mass and capacity sequence. The Vertical Lemma and the Inert Principle allow us to determine whether the intersection point is light or dark. ♠

Corollary 2.3 *Suppose that we know how the plaid polygons intersect a single $\mathcal{H}$ line inside Σ. Then the intersection of the plaid polygons with Σ is determined by the signs of the mass and capacity sequences.*

Proof: We will suppose that we know how the plaid polygons intersect the bottom edge of Σ. The case for any other edge has a similar treatment. Let Q be some unit square in Σ for which we have not yet determined the plaid model inside Q. We can take Q to be as low as possible. But then we know how the plaid model intersects the bottom edge of Q, and the signs of the mass and capacity sequences determine how the tiling intersects the left and right edges. But then the Fundamental Theorem for the plaid model tells us that ∂Q intersects the plaid polygons in either 0 or 2 points. This allows us to determine how the plaid polygons intersect the top edge of Q. ♠

2.5 A Matching Criterion

We mean to define all the objects in the previous section with respect to parameter p/q and p'/q'. Our notation convention will be that the object X corresponds to p/q whenever the same kind of object X' corresponds to p'/q'.

Let Π denote the union of plaid polygons with respect to p/q and let Π' denote the union of plaid polygons with respect to p'/q'. Suppose that Σ and Σ' are rectangles that are equivalent via a vertical translation Υ. That is, Υ preserves the y-axis and $\Upsilon(\Sigma') = \Sigma$. To be concrete, say that

$$\Upsilon(x, y) = (x, y + \xi). \tag{10}$$

We will give a criterion which guarantees that

$$\Upsilon(\Sigma' \cap \Pi') = \Sigma \cap \Pi. \tag{11}$$

Arithmetic Alignment: A necessary condition for Equation 11 is that the signs of $\{c_j\}$ are the same as the corresponding signs of $\{c'_j\}$ and the signs of $\{m_j\}$ are the same as the corresponding signs of $\{m'_j\}$. More precisely,

$$\mathrm{sign}(c'_j) = \mathrm{sign}(c_{j+\xi}), \qquad \mathrm{sign}(m'_j) = \mathrm{sign}(m_{j+\xi}). \tag{12}$$

We say that (Σ, Π) and (Σ', Π') are *arithmetically aligned* of Equation 12 holds for all relevant indices.

It seems plausible that arithmetic alignment is sufficient for Equation 11 to hold, but we don't have a proof. We need more ingredients to make things work out cleanly.

Geometric Alignment: There is a natural correspondence between the vertical intersection points in Σ and the vertical intersection points in Σ'. Let z' be a vertical intersection point in Σ. Let $\{i', j'\}$ be the pair of indices so that the slanting lines through $(0, i')$ and $(0, j')$ contain z'. We let z denote the intersection of the slanting lines, of the same type, through $(0, i)$ and $(0, j)$. Here $i = i' + \xi$ and $j = j' + \xi$. We say that z' and z are *geometrically aligned* of $\Upsilon(z')$ and z are contained in the same unit vertical segment of Σ. We say that (Σ, Π) and (Σ', Π') are *geometrically aligned* if z and z' are geometrically aligned for every vertical intersection point $z' \in \Sigma'$.

It seems very likely that arithmetic and geometric alignment together imply Equation 11 but we don't have a proof. We need one more small ingredient.

Weak Horizontal Alignment: We say that (Σ, Π) and (Σ', Π') are *weakly horizontally aligned* if there are $\mathcal{H}$ lines H' and H such that

$$\Upsilon(\Sigma \cap H \cap P) = \Sigma' \cap H' \cap \Pi'. \tag{13}$$

In other words, the tilings look the same on a single horizontal segment. This is exactly the criterion which appears in Corollary 2.3.

Now we come to our Matching Criterion.

Lemma 2.4 (Matching Criterion) *Suppose that*

- (Σ, Π) *and* (Σ', Π') *are weakly horizontally aligned.*
- (Σ, Π) *and* (Σ', Π') *are geometrically aligned.*
- (Σ, Π) *and* (Σ', Π') *are arithmetically aligned.*

Then $\Upsilon(\Sigma' \cap \Pi') = \Sigma \cap \Pi$.

Proof: Given Corollary 2.3, this is practically a tautology. The procedure given in Corollary 2.3 assigns exactly the same tiles to $\Sigma \cap \Pi$ as it does to $\Upsilon(\Sigma' \cap \Pi')$. ♠

2.6 The Big Polygon

We define τ to be the unique integer solution in $(0, \omega/2)$ to

$$2p\tau \equiv \pm 1 \mod \omega. \tag{14}$$

We call τ the *tune* of the parameter p/q.

Lemma 2.5 *For* $k = 0, ..., (\omega - 1)/2$*, the lines of capacity* $2k$ *have the form* $x = \pm k\tau$ *and* $y = \pm k\tau$*. For* $k = 1, 3, ..., (\omega - 1)$*, the the lines of mass* k *have* y*-intercepts* $(0, \pm k\tau)$*. These equations are all taken mod* ω*.*

Theorem 2.6 *Let* B *be any block. For each even* $k \in [0, p+q]$ *there are* 2 *lines in* $\mathcal{H}$ *and* 2 *lines in* $\mathcal{V}$ *which have capacity* k *and intersect* B*. Each such line carries* k *light points in* B*.*

Let p/q be an even rational parameter and let $\omega = p + q$. We define the x-diameter of a set to be the diameter of its projection onto the x-axis. Here is a result repeated from [**S0**].

Theorem 2.7 *Let B denote the first block. Then there exists a plaid polygon in B whose x-diameter is at least $\omega^2/(2q) - 1$. Moreover, this polygon has bilateral symmetry with respect to reflection in the horizontal midline of B.*

Proof: Let L be the horizontal line of capacity 2 and positive sign which intersects B. Let $z_1 = (0, y) \in L$. By Lemma 2.5, we know that z_1 is a light point of mass 1. Let $z_2 = (\omega^2/2q, y)$. We compute that z_2 is another light point on L. Since L has capacity 2, these are the only two light points on L. The lattice polygon which crosses the unit horizontal segment containing z_1 must also cross the unit horizontal segment containing z_2 because it has to intersect $L \cap B$ twice. This gives the lower bound on the x-diameter.

Let Γ' denote the reflection of Γ in the horizontal midline of B. We want to show that $\Gamma' = \Gamma$. Let V_1 and V_2 denote the two vertical lines of B having capacity 2. These lines are symmetrically placed with respect to the vertical midline of B. Hence, one of the two lines, say V_1, lies less than $\omega/2$ units away from the y-axis. Since $\omega/2 < \omega^2/(2q)$, the point z_2 is separated from the y-axis by V_1. Hence both Γ and Γ' intersect V_1. Since there can be at most 1 plaid polygon which intersects $V_1 \cap B$, we must have $\Gamma = \Gamma'$. ♠

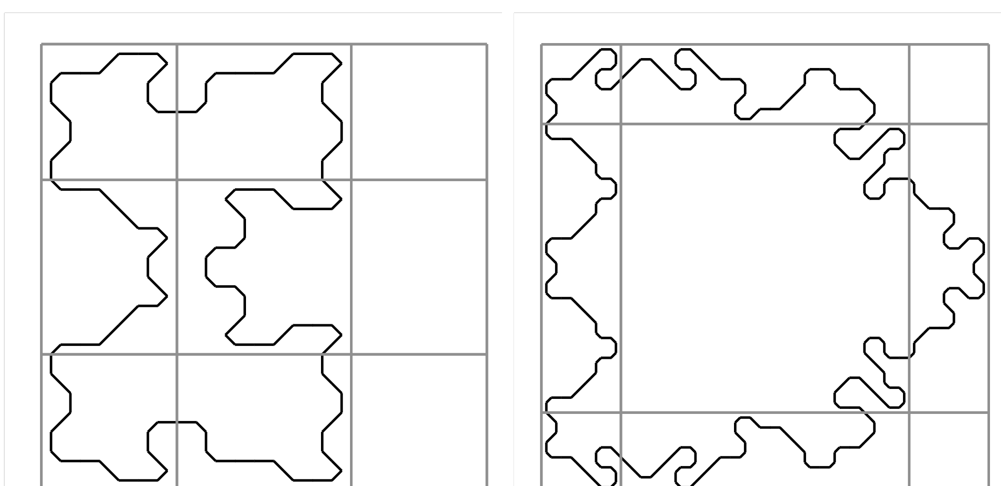

Figure 2.1: The polygon Γ for parameters 5/18 and 14/31.

We call Γ *the big polygon*. Figure 2.1 shows a picture for the 2 somewhat arbitrarily chosen parameters. The lines of capacity 2 are also shown in the figure.

We define

$$\langle\Gamma\rangle = \Gamma \cap (\{1/2\} \times \boldsymbol{Z}). \tag{15}$$

We call points in $\langle\Gamma\rangle$ the *anchor points* of Γ. We know from our proof that Γ contains at least 2 anchor points, namely

$$z_1 + (1/2, 0), \qquad z_2 + (1/2, 0). \tag{16}$$

The idea behind our proof of the Unbounded Orbits Theorem is to take a geometric limit of a sequence of these polygons corresponding to a sequence $\{p_n/q_n\}$ of even rationals converging to some irrational $A \in (0,1)$. As we will explain in §3.7, this argument must be done carefully in order to work. The key step in our proof is understanding the set $\langle\Gamma\rangle$ well. We will understand this set in a recursive way, using the Matching Criterion to understand what happens for complicated rational parameters in terms of simpler ones which are close in the Diophantine sense.

3 The PET Description

The material here is a compressed account of [**S0**,§3] and [**S0**,§8]. The main difference is that the objects we called $\widehat{\Xi}$ and $\widehat{X}$ will here be called Ξ and X. This will simplify the notation.

3.1 The Classifying Space

We fix $P \in [0,1]$ and let Λ_P denote the lattice generated by the vectors

$$(0,4,2P,2P), \qquad (0,0,2,0), \qquad (0,0,0,2). \tag{17}$$

This lattice acts on $\boldsymbol{R}^4$, but the action on the first coordinate is trivial.

We take the quotient

$$X_P = (\{P\} \times \boldsymbol{R}^3)/\Lambda_P. \tag{18}$$

The cube $\{P\} \times [-2,2] \times [-1,1]^2$ serves as a fundamental domain for the action of Λ_P on $\{P\} \times \boldsymbol{R}^3$. However, the boundary identifications depend on P. The *total space* is the union

$$X = \bigcup_{P \in [0,1]} X_P \tag{19}$$

This space is a flat affine manifold: It is realized as a quotient of the form

$$([0,1] \times \boldsymbol{R}^3)/\Lambda, \tag{20}$$

where Λ is the abelian group of affine transformations generated by

1. $T_1(x_0,x_1,x_2,x_3) = (x_0, x_1+4, x_2+4x_0, x_3+4x_0)$.
2. $T_2(x_0,x_1,x_2,x_3) = (x_0, x_1, x_2+2, x_3)$.
3. $T_3(x_0,x_1,x_2,x_3) = (x_0, x_1, x_2, x_3+2)$.

The set $[0,1] \times [-2,2] \times [-1,1]^2$ serves as a fundamental domain for X.

3.2 The Partition

We have a partition of X_P into $13 = 12 + 1$ regions. One of the regions is labeled by an "empty symbol" and the other 12 regions are labeled by ordered pairs of elements in $\{N, S, E, W\}$. For each tile label ξ, let $X_P(\xi)$ denote the piece of the partition labeled by ξ.

We describe the partition exactly in [**S2**,§3]. Here we will just describe the two relevant features. First, for each of the 13 choices of ξ, the *total piece*

$$X(\xi) = \bigcup_{P \in [0,1]} X_P(\xi) \tag{21}$$

is a union of finitely many convex polytopes whose vertices have integer coordinates.

To describe the second feature, we introduce coordinates (P, T, U_1, U_2) on $\boldsymbol{R}^4$. We have a fibration which maps (P, T, U_1, U_2) to (P, T). The fibers are called the (U_1, U_2) fibers. The second feature is that the partition intersects each (U_1, U_2) fiber generically in a 4×4 grid of rectangles. The rectangles are parallel to the coordinate axes. Figure 3.1 shows what we have in mind. The squares labeled N,E,W,S are really labeled by the empty symbol in our scheme. However, we put these letters in them do indicate the scheme for filling in the rest of the labels.

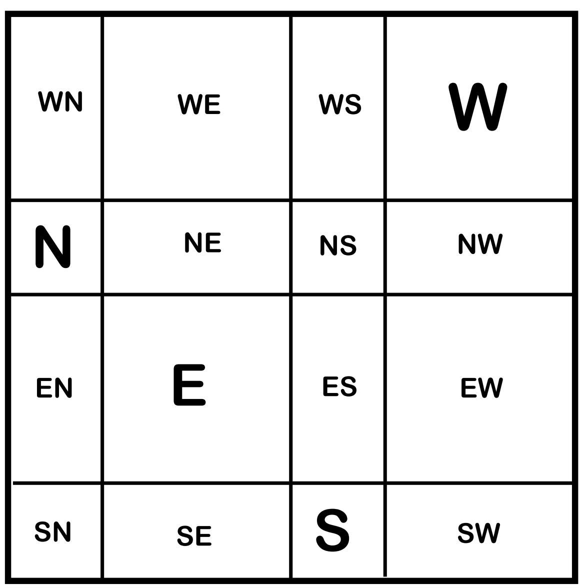

Figure 3.1: The intersection of the partition with a fiber.

3.3 The Classifying Map

For each $P \in (0,1)$ we have a map

$$\Xi_P : \boldsymbol{R}^2 \to X_P \tag{22}$$

defined by

$$\Xi(x,y) = (P, 2Px + 2y, 2Px, 2Px + 2Py) \mod \Lambda_P. \tag{23}$$

For each even rational parameter p/q we set $P = 2p/(p+q)$, as above, and restrict Ξ_P to the set $\mathcal{C}$ of centers of square tiles. In [**S0**] we proved that $\Xi_P(\mathcal{C})$ never hits a wall of a partition in the rational case.

For fixed p/q, we define a tiling of $\boldsymbol{R}^2$ as follows. For each $c \in \mathcal{C}$ we see which partition piece contains the point Ξ, and then we draw the corresponding tile in the unit square centered at c. If $\Xi_P(c)$ lands in the tile labeled by the empty symbol, then we draw nothing in the tile.

The PET Equivalence Theorem in [**S0**] says that the tiling we get agrees with the tiling from the plaid model, and moreover the orientations on the tiles are cpherent. That is, for each edge involved in two connectors, one of the connectors points into the edge and the other points out. That is, each of the plaid polygons has a coherent orientation.

The next result says something about the geometry of the clasifying map. The result holds for any half integer, but we only care about the case of 1/2.

Lemma 3.1 *If $Y = \{1/2\} \times \boldsymbol{Z}$ then $\Xi_P(Y)$ is contained in a single geodesic γ_P in the fiber of Ξ_P over the point (P,P). Moreover, γ_P has slope -1 when developed into the plane so that the (U_1, U_2) coordinate axes are identified with the coordinate axes of the plane.*

Proof: To see this, we compute

$$\Xi_P(1/2, m + 1/2) = (P, P + 2m, P, P + 2Pm) \mod \Lambda_P.$$

Subtracting off the vector $(2m, Pm, Pm)$, we get

$$\Xi_P(1/2, m + 1/2) = (P,P,P,P) + (0,0,-mP,mP) \mod \Lambda_P. \tag{24}$$

As m varies, the image remains on the geodesic of slope -1 through the point (P,P,P,P). At the same time the walls of the partition intersect the fiber in horizontal and vertical lines. ♠

3.4 The PET Interpretation

Now we see how to interpret the space X_P as a PET

For each parameter p/q, we set $P = 2p/(p+q)$, and our space is X_P. The partition is given by

$$X_{S\downarrow} \cup X_{W\leftarrow} \cup X_{N\uparrow} \cup X_{E\rightarrow}. \cup X_{\Box}. \tag{25}$$

The remaining set $X_{\Box}$ is just the complement. These sets have an obvious meaning. For instance, $X_{E,\rightarrow}$ is the union of regions which assign tiles which point into their east edges. And so on. The second partition of X is obtained by reversing all the arrows. So, the first partition is obtained by grouping together all the regions which assign tiles which point into a given edge, and the second partition is obtained by grouping together all the regions which assign tiles which point out of a given edge.

We have 5 *curve following maps* $\Upsilon_{\Box} : X_{\Box} \to X_{\Box}$ is the identity, and then $\Upsilon_S : X_{S\downarrow} \to X_{N\downarrow}$. And so on. These maps are all translations on their domain. For instance, Υ_P has the following property:

$$\Upsilon_S \circ \Xi(x,y) = \Xi(x, y-1). \tag{26}$$

From Equation 23, we get

$$\Upsilon_S(P,T,U_1,U_2) = (P, T-2, U_1, U_2 - 2P) \mod \Lambda'. \tag{27}$$

There is a natural map on $\mathcal{C}$, the set of centers of the square tiles. We simply follow the directed edge of the tile centered at c and arrive at the next tile center. By construction, Ξ_P conjugates this map on $\mathcal{C}$ to our polyhedron exchange transformation. By construction Ξ_P sets up a dynamics-respecting bijection between certain orbits in X_P and the plaid polygons contained in the fundamental domain, with respect to the parameter p/q.

Now we explain what we mean by vector dynamics. We assign $(0,-1)$ to the region $X_{S\downarrow}$, and $(-1,0)$ to $X_{W\leftarrow}$, etc. When we follow the orbit of $\Xi(c)$, we get record the list of vectors labeling the regions successively visited by the point. This gives vectors v_1, v_2, v_3. The vectors c, $c+v_1$, $c+v_1+v_2$, etc. are the vertices of the plaid polygon containing c. This is what we mean by saying that the plaid polygons describe the vector dynamics of certain orbits of the PET.

3.5 The Irrational Case

Let $\mathcal{C}$ denote the set of centers of the square tiles. The classifying pair (Ξ_P, X_P) makes sense even when P is irrational. However, Ξ_P might map points of $\mathcal{C}$ into the boundary of the partition. For instance

$$\Xi_P(1/2, 1/2) = (P, P+1, P, 2P) \tag{28}$$

and this point lies in the wall of the partition.

To remedy this situation, we introduce an *offset*, namely a vector $V \in \boldsymbol{R}^3$, and we define

$$\Xi_{P,V} = \Xi_P + (0, V). \tag{29}$$

(The first coordinate, namely P, does not change.) Since Λ_P acts on $\boldsymbol{R}^3$ with compact quotient, we can take V to lie in a compact subset of $\boldsymbol{R}^3$ and still we will achieve every possible map of this form. We call V a *good offset* if $\Xi_{P,V}(\mathcal{C})$ is disjoint from the walls of the partition. In this case, the PET dynamics permutes the points of $\Xi_{P,V}(\mathcal{C})$ and every orbit of one of these points is well defined.

Here we give a criterion for V to be a good offset. Define

$$\boldsymbol{Q}[P] = \{r_1 + r_2 P |\ r_1, r_2 \in \boldsymbol{Q}\}. \tag{30}$$

Lemma 3.2 *Suppose $V = (V_1, V_2, V_3)$ is such that*

$$V_1 \in \boldsymbol{Q}[P], \qquad V_1, V_2 \not\in \boldsymbol{Q}[P].$$

Then V is a good offset.

Proof: Here is an immediate consequence the two main properties of our partition. Suppose F is a (U_1, U_2) fiber over a point $(P, T) \in \boldsymbol{Q}[P] \times \boldsymbol{Q}[P]$. Then the walls of the partition intersect F in rectangles bounded by lines defined over $\boldsymbol{Q}[P]$. That is, the lines have the form $x = x_0$ and $y = y_0$ for various choices of $x_0, y_0 \in \boldsymbol{Q}[P]$. So, if we have a point (P, T, U_1, U_2) with $T \in \boldsymbol{Q}[P]$ and $U_1, U_2 \not\in \boldsymbol{Q}[P]$, then the point lies in the interior of some partition piece. Given the formula in Equation 23, the map $\Xi_{P,V}$ has this property. ♠

3.6 Geometric Limits

Suppose now that $\{p_k/q_k\}$ is a sequence of even rational parameters converging to some irrational parameter A. Let

$$P_k = \frac{2p_k}{p_k+q_k}, \qquad P = \frac{2A}{1+A}. \tag{31}$$

By definition $P_k \to P$. Likewise, the classifying maps Ξ_{P_k} converge, uniformly on compact sets, to Ξ_P. As we have already mentioned, the map Ξ_P is not really a very good map. Here we explain a more flexible kind of limit we can take.

For ease of notation, we set $\Xi_{P_k} = \Xi_k$. Let $Z = \{z_k\}$ be a sequence of points in $\boldsymbol{Z}^2$. We define

$$\Xi_k^Z(c) = \Xi_k(c+z_k). \tag{32}$$

In other words, we compose by a translation, so that the action of Ξ_k^Z around the origin looks like the action of Ξ_k near z_k.

Lemma 3.3 *The sequence $\{\Xi_k^Z\}$ converges on a subsequence to $\Xi_{P,V}$ for some offset vector V.*

Proof: Since Ξ is a locally affine map for each parameter, there is some vector V_k so that

$$\Xi_k^Z = \Xi_k + V_k.$$

We can always take V_k to lie within a compact set, namely

$$\{0\} \times [-2,2] \times [-1,1]^2,$$

which is just a translated copy of the fundamental domain for the lattice Λ_{P_k}. (It is translated so that the first coordinate is 0.) But now we can take a subsequence so that the sequence $\{V_k\}$ converges to some vector V. By construction, $\{\Xi_k^Z\}$ converges to $\Xi_{P,V}$. ♠

The vector V depends somehow on the sequence Z. We call Z a *good sequence* if the vector V is a good offset.

3.7 A Failed Attempt

To explain the subtlety in the proof of the Unbounded Orbits Theorem, we make a failed attempt at proving it.

We choose some sequence $\{p_k/q_k\}$ which converges to A and let $\{\Gamma_k\}$ be the corresponding sequence of big polygons. Let z_k be a point of $\mathcal{C}$ uniformly close to one of the two points of $\langle\Gamma_k\rangle$ that we know about already, namely those in Equation 16. Let $Z = \{z_k\}$.

Now, $\Gamma_k - z_k$ intersects a tile centered at a point c_k which is uniformly close to the origin. Hence, on a subsequence, the polygons $\{\Gamma_k - z_k\}$ converge to an infinite polygonal path Γ_∞. If Z is a good sequence, then $\Xi_{P,V} = \lim \Xi_k^Z$ has an infinite orbit whose vector dynamics produces Γ_∞. We will discuss this point in more detail in §4.6.

One problem with this construction is that we don't know that both coordinate projections of Γ_∞ have infinite diameter. A more serious problem is that Z is not a good sequence. A calculation, which we omit, shows that Ξ_P maps the points $z_j + (\pm 1/2, \pm 1/2)$ into the walls of the partition.

In making an argument which works, we want to keep the idea of choosing points near $\langle\Gamma_k\rangle$, but we need to know that this set has many more points before we can make fundamentally different choices.

4 The Main Argument

4.1 Approximating Irrationals

Our goal in this section is to define a sequence $\{p_n/q_n\}$ of even rationals which converges to an irrational $A \in (0,1)$ and has good Diophantine properties. This sequence is related to, but usually different from, the sequence of continued fraction approximants. We defer the proofs of the technical lemmas to the next chapter.

Even Predecessors: Two rationals a_1/b_1 and a_2/b_2 are called *Farey related* if $|a_1b_2 - a_2b_1| = 1$. Let p'/q' and p/q be two even rational paramters. We write $p'/q' \leftarrow p/q$ if p/q and p'/q' are Farey related and $\omega' < \omega$. Here $\omega = p + q$ as usual. We call p'/q' the *even predecessor* of p/q. This rational is unique.

Core Predecessors: Recall that τ is the tune of p/q, defined in §2.6. We define $\kappa \geq 0$ to be the integer so that

$$\frac{n}{2n+1} \leq \frac{\tau}{\omega} < \frac{n+1}{2(n+1)+1}. \tag{33}$$

We only get equality on the left hand side when $p/q = 1/2n$. Define

$$\widehat{p} = p - 2\kappa p', \qquad \widehat{q} = q - 2\kappa q'. \tag{34}$$

Statement 1 of Lemma 5.1 proves that $\widehat{p}/\widehat{q}$ is an even rational in $(0,1)$. We call $\widehat{p}/\widehat{q}$ the *core predecessor*.

Predecessors: Given an even rational parameter p/q, we define the *predecessor* p^*/q^* as follows:

- If $p = 1$ then $p^*/q^* = 0/1$.
- If $p \geq 2$ and $\kappa = 0$ then $p^*/q^* = p'/q'$, the even predecessor of p/q.
- If $p \geq 2$ and $\kappa \geq 1$ then $p^*/q^* = \widehat{p}/\widehat{q}$, the core predecessor of p/q.

We write $p^*/q^* \prec p/q$. This definition turns out to be very well adapted to the plaid model. Lemma 5.1 collects together many of the relations between these rationals.

The Predecessor Sequence: In §5 we will prove the following result.

Lemma 4.1 *Let $A \in (0,1)$ be irrational. Then there exists a sequence $\{p_k/q_k\}$ such that*

- $p_0/q_0 = 0/1$
- $p_k/q_k \prec p_{k+1}/q_{k+1}$ *for all* k
- $A = \lim p_k/q_k$.

We call $\{p_k/q_k\}$ the *predecessor sequence*. This terminology suggests that the predecessor sequence is unique. However, we will not bother to prove this. We just need existence, not uniqueness.

Diophantine Result: Let $\{p_k/q_k\}$ be the predecessor sequence converging to A. We classify a term p_k/q_k in the predecessor sequence as follows:

- weak: $\tau_{k+1} < \omega_{k+1}/4$. Here $\kappa_{k+1} = 0$.
- strong: $\tau_{k+1} \in (\omega_{k+1}/4, \omega_{k+1}/3)$. Here $\kappa_{k+1} = 0$.
- core: $\tau_{k+1} > \omega_{k+1}/3$. Here $\kappa_{k+1} \geq 1$.

Lemma 4.2 *The predecessor sequence has infinitely many non-weak terms. For each non-weak term p_k/q_k, we have*

$$\left|A - \frac{p_k}{q_k}\right| < \frac{48}{q_k^2}.$$

The Approximating Sequence: We define the approximating sequence to be the set of terms p_k/q_k in the predecessor sequence such that either

- p_k/q_k is core.
- p_k/q_k is strong and p_{k-1}/q_{k-1} is not core.

If there are infinitely many core terms in the predecessor sequence, then the approximating sequence contains all of these. If there are only finitely many core terms in the predecessor sequence, then there are infinitely many strong terms, and the approximating sequence contains all but finitely many of these. So, in all cases, the approximating sequence is an infinite sequence.

4.2 Arc Copying

Let $R_{p/q}$ denote the rectangle bounded by the bottom, left, and top of the first block, and by whichever vertical line of capacity at most 4 is closest to the left edge of the first block. Let $\gamma_{p/q}$ denote the subset of $\Gamma_{p/q}$ contained in the box $R_{p/q}$. When the dependence on the parameter is implied we will suppress it from our notation.

Figure 4.1 shows the big polygons associated to two different parameters. Notice that the polygon $\Gamma_{12/29}$ copies some of $\Gamma_{5/12}$. The boxes $B_{5/12}$ and $B_{12/29}$ are the first columns (i.e. the union of the leftmost 4 sub-rectangles) of the tic-tac-toe grids shown in each of the two pictures.

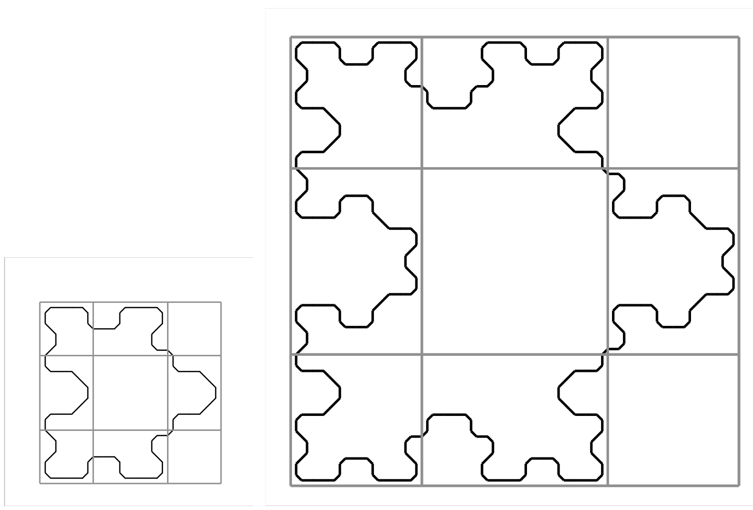

Figure 4.1: Arc copying for 5/12 and 12/29.

Let TH and BH denote the top and bottom horizontal lines of capacity 2 with respect to some rational parameter. In the statement of the results below, it will be clear which parameters these lines depend on.

Lemma 4.3 (Box) *For any even rational parameter p/q, the set $\gamma_{p/q}$ is an arc whose endpoints lie on the right edge of $R_{p/q}$.*

Theorem 4.4 (Copy) *Let $p_0/q_0, p_1/q_1$ be two successive terms in the approximating sequence. Then there is some vertical translation Υ such that that $\Upsilon(R_0)$ is contained below the horizontal midline of R_1, and $T(\gamma_0) \subset \gamma_1$. Moreover, either $\Upsilon(BH_0) = BH_1$ or $\Upsilon(TH_0) = BH_1$.*

4.3 Marked Boxes

Now we will somewhat abstract the main features of the pair $(R_{p/q}, \gamma_{p/q})$, even though approximatingly these are the sets we will be considering.

Say that a *marked box* is a pair $\beta = (R, \gamma)$

- R is a rectangle with sides parallel to the coordinate axes, whose left edge is contained in the y axis.

- γ is a polygonal path which has both endpoints on the right edge of R and contains two points of the form $(1/2, y)$, one strictly in the lower half of R and one strictly in the upper half.

Given two marked boxes $\beta_j = (R_j, \gamma_j)$, we write $\beta_1 \prec \beta_2$ if $R_1 \subset R_2$ and $\gamma_1 \subset \gamma_2$. We insist that the side lengths of R_2 are at least one unit longer than the corresponding side lengths of R_1. This is the kind of copying produced by the Copy Theorem.

Now we explore the consequences of iterating the result of the Copy Theorem. Note that the Copy Theorem really says that the arc γ contains two translated copies of $\widehat{\gamma}$, one above the horizontal midline of R and one below. This follows from the bilateral symmetry of γ and of $\widehat{\gamma}$. So, when we iterate the Copy Theorem, we should expect the emergence of a tree-like structure.

Let T be a directed binary tree with no forward infinite path. If T is finite, then T has one initial vertices $2^k - 1$ vertices in total. If T is infinite, then T has no initial vertex. Given any vertex v of T, we define $\wedge v$ to be the subtree of T whose initial node is v. This is always a finite binary tree.

We say that a collection of marked boxes *realizes* T if there is map Φ from the set of vertices of T to the set of marked boxes, such that

- If $v \leftarrow w$ then $\Phi(v) \prec \Phi(w)$.

- If $v_j \leftarrow w$ for $j = 1, 2$, then $\Phi(v_1)$ and $\Phi(v_2)$ are disjoint and the restriction of Φ to $\wedge v_1$ equals the restriction of Φ to $\wedge v_2$ up to a vertical translation τ_w. We call τ_w the *translation associated to w*.

Suppose we have a finite string $p_1/q_1, ..., p_n/q_n$ of parameters in the approximating sequence. The Box Lemma says that all the pairs (R_k, γ_k) are marked boxes. Iterated application of the Copy Theorem, and bilateral symmetry, gives us a realization of the directed binary tree with $2^n - 1$ vertices.

Figure 4.2 shows a schematic picture of the case when $n = 3$.

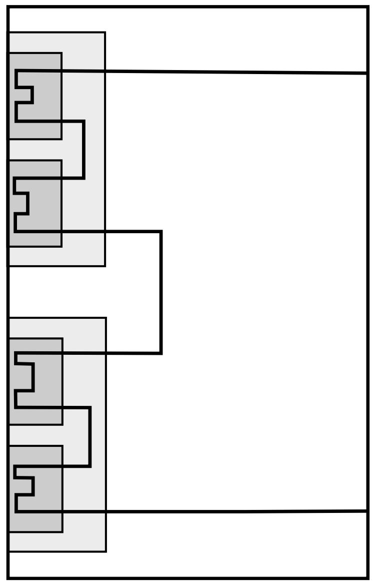

Figure 4.2: A pattern of boxes realizing a tree.

Figure 4.3 shows a particular planar embedding of an infinite binary directed tree. The bottom path lies in the x-axis. The terminal nodes of the tree lie on the y-axis.

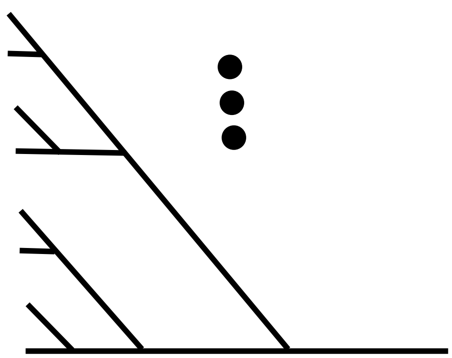

Figure 4.3: A special infinite binary tree.

Let Φ be a realization of the tree T suggested by Figure 4.3. We call Φ *normalized* if Φ maps any vertex on the bottom path of T to a box whose curve contains $(1/2, 0)$, and if the boxes obtained in this way are beneath any other boxes associated to Φ. This is to say that all the translations τ_w considered in the definition of Φ are positive translations along the x-axis.

4.4 The Realization Lemma

Let T be an infinite directed binary tree, as above.

Lemma 4.5 (Realization) *Let $A \in (0,1)$ be irrational. Let $\{p_n/q_n\}$ be the approximating sequence associated to A. Then there exists a normalized tree realization Φ of the infinite binary tree T with the following property. For each n there is a vertex v_n of T such that $\Phi(v_n)$ is translation equivalent to the marked box β_{p_n/q_n}.*

Proof: Looking at the finite sequence $p_1/q_1, ..., p_n/q_n$ of parameters in the approximating sequence, we get a realization Φ_n of the tree T_n having $2^n - 1$ vertices. We can adjust Φ_n by a vertical translation so that the bottom distinguished point of Φ_n is $(1/2, 0)$. By construction, the bottom half of Φ_{n+1} coincides with Φ_n. This means that we can take a geometric limit and get a normalized realization of the infinite tree. The vertex sequence $\{v_n\}$ is just the sequence of vertices along the x-axis in the tree shown in Figure 4.3. ♠

Let d_k be the translation length of the translation τ_{v_k} associated to the realization Φ. Recall that $[t]_2$ denotes the value of t mod $2\boldsymbol{Z}$ which lies in $[-1, 1)$. Let $P = 2A/(1 + A)$.

Lemma 4.6 $\lim_{k\to\infty}[Pd_k]_2 = 0$.

Proof: Let $t_k = t(p_k/q_k)$. Let $\eta_k = \omega_k - 2t_k$. Geometrically, η_k is the distance between the two horizontal lines of capacity 2 associated to p_k/q_k. It follows from the last Statement of the Copy Theorem that $d_k = \eta_k \pm \eta_{k-1}$. Thus, it suffices to prove that $\lim_{k\to\infty}[P\eta_k]_2 = 0$.

Let $P_k = 2p_k/(p_k + q_k)$. The function $y \to [P_k y]_2$ assigns value of $\pm 1/\omega_k$ to points on the two horizontal lines of capacity 2. Hence

$$\lim_{k\to\infty}[P_k\eta_k]_2 \to 0. \tag{35}$$

$$|(P - P_k)\eta_k| \le |P - P_k|\omega_k < 96/\omega_k. \tag{36}$$

The last inequality comes from Lemma 4.2 and the fact that the map $A \to P = 2A/(A + 1)$ is 2-lipschitz. This lemma now follows from the triangle inequality and Equations 35 and 36. ♠

4.5 Taking Many Limits

Let Φ be the normalized tree realization from The Realization Lemma. Let $\{d_k\}$ be the sequence of translations associated to Φ, as in Lemma 4.6. Let $B=\{\epsilon_k\}$ be any infinite binary sequence. Define

$$\Phi_k=\Phi-(0,y_k), \qquad y_k=\sum_{i=1}^{k}\epsilon_i d_i. \tag{37}$$

Note that $(0,1/2)$ remains a vertex of some of the curves associated to Φ_k.

Lemma 4.7 *For any positive R, the images of Φ_k and Φ_{k+1} agree inside the ball of radius R about 0 for all sufficiently large k.*

Proof: The realizations Φ_k and Φ_{k+1} either agree, or there are three vertices v_1, v_2, w such that

- $v_1, v_2 \leftarrow w$ and v_1 lies below v_2.
- Both $\Phi_k(w)$ and $\Phi_k(v_1)$ contain $(0,1/2)$.
- $\Phi_{k+1}=-\tau_w\circ\Phi_k$.

Moreover, the vertex w is combinatorially k steps from a terminal vertex. We have

$$\Phi_{k+1}(\wedge v_2)=-\tau_w\Phi_k(\wedge v_2)=\Phi_k(\wedge v_1).$$

Once k is sufficiently large, $\Phi_k(\wedge v_1)$ and $\Phi_{k+1}(\wedge v_2)$ account for all the boxes associated to these realizations which come within R of the origin. ♠

Lemma 4.7 allows us to define the limit $\Phi_\infty=\lim_k\Phi_k$. By construction, Φ_∞ is another realization of an infinite tree. The curve associated to Γ_∞ – namely, the union of all the curves associated to all the boxes – has infinite diameter projections onto both coordinate axes.

Let y_k be as in Equation 37. Let $\Xi_{P,V}$ be the a limit of the translated classifying maps

$$\Xi_P-(0,y_k). \tag{38}$$

This definition depends on the binary sequence $B=\{\epsilon_k\}$. We call B a *good sequence* if V is a good offset. As we have already mentioned, one way to guarantee this condition is to arrange that $\Xi_{P,V}(1/2,1/2)$ has coordinates of the form (P,T,U_1,U_2) with $T\in\boldsymbol{Q}[P]$ and $U_1,U_2\not\in\boldsymbol{Q}[P]$.

Lemma 4.8 *There is a good binary sequence relative to* Φ.

Proof: Let Y and Ξ_P and γ_P be the sets from Lemma 3.1. Since γ_P has slope -1 when developed out into the plane, the geodesic γ_P only intersects the set

$$(\boldsymbol{Q}[P] \times \boldsymbol{R}) \cup (\boldsymbol{R} \times \boldsymbol{Q}[P]) \tag{39}$$

in a countable collection of points.

Let $Y_D \subset Y$ denote the set of all points of the form

$$\Big(\frac{1}{2}, \sum_{i=1}^{k} \epsilon_i d_i\Big). \tag{40}$$

Here the sum ranges over all finite binary sequences, and all $k \in \boldsymbol{N}$. To prove our result, it suffices to prove that $\Xi_P(Y_D)$ has uncountably many accumulation points contained inside γ_P. Since γ_P is dense in the fiber containing it, we have to take care that the accumulation points we find all belong to γ_P.

Now we bring in the information given by Lemma 4.6. Since we can automatically set some of the terms in our sequence to zero, it suffices to prove our result for any subsequence of D we like. So, we can assume without loss of generality that

$$|[Pd_k]_2| < 10^{-k}. \tag{41}$$

Passing to a further subsequence, we can reduce to the case where either $[Pd_k]_2$ is always positive or always negative. We will consider the positive case. The negative case has the same treatment, except for a sign. Let $\delta_k = [Pd_k]_2$. We have $\delta_k \in (0, 10^{-k})$.

Looking at Equation 24, we see that $\Xi(Y_D)$ develops into a bounded subset of the plane. More precisely, $\Xi(Y_D)$ develops into the plane as a translate of the set $C' \times (-C')$, where C' is the set of all finite sums of the form

$$\sum_{i=1}^{n} \epsilon_k \delta_k, \qquad \epsilon_k \in \{0,1\}. \tag{42}$$

But C' is dense in a Cantor set which consists of all the infinite sums of same form. Hence the set of accumulation points of $\Xi(Y_\mu)$ contains a Cantor set, all of whose points belong to γ_P. ♠

4.6 The End of the Proof

Let $P = 2A/(1+A)$ as usual. Let $\mathcal{C}$ be the set of centers of the square tiles. There exists some binary sequence B and some associated limit realization Φ_∞ such that

- The corresponding classifying map $\Xi_{P,V}$ carries every point of $\mathcal{C}$ into the interior of a partition piece. That is, V is a good offset. Let $\Xi_\infty = \Xi_{P,V}$. This map is the limit of maps $\Xi_n - (0, y_n)$. Here y_n is such that $y_n + (0, 1/2) \in \langle \Gamma_n \rangle$.

- The curve Γ_∞ associated to Φ_∞ has unbounded projections in both coordinate directions.

- Γ_∞ is the Hausdorff limit of the curves $\gamma_n - (0, y_n)$, and this is the same as the Hausdorff limit of the loops $\Gamma_n - (0, y_n)$. The portion of $(\Gamma_n - \gamma_n) - (0, y_n)$ exits every compact set in the plane and is not seen in the limit.

Let $c_0 = (1/2, 1/2)$. This point belongs to Γ_∞. To finish the proof, we have to check that Γ_∞ realizes the vector dynamics for Ξ_∞ associated to the point c_0. Let B_R denote the ball of radius R about the origin.

Note that $\Gamma_n - (0, y_n)$ describes the vector dynamics of $\Xi_n - (0, y_n)$ associated to c_0. Since Ξ_∞ is defined on all of $\mathcal{C}$, we get the following result by continuity. For any $R > 0$ there is some N such that $n > N$ implies that $\Xi_n - (0, y_n)$ and Ξ_∞ have the same vector dynamics for relative to all points of $\mathcal{C} \cap B_R$. That is, the partition labels assigned to points of $\mathcal{C} \cap B_R$ by the two maps are the same. Hence, Γ_∞ describes the vector dynamics of Ξ_∞ associated to c_0 on arbitrarily large balls about the origin. This is the same as saying that Γ_∞ describes the vector dynamics of Ξ_∞ associated to c_0.

This completes the proof of the Unbounded Orbits Theorem modulo the results quoted in §4.1, the Box Lemma, and the Copy Theorem.

5 Some Elementary Number Theory

In this chapter we will justify all the statements quoted in §4.1. We will also prove a number of other technical results which establish useful identities and inequalities between the rationals discussed in §4.1.

5.1 All About Predecessors

We fix an even rational parameter p/q. To avoid trivialities, we assume that $p > 1$. Given an even rational parameter p/q, we make the following definitions:

- p'/q' is the even predecessor of p/q.
- $\widehat{p}/\widehat{q}$ is the core predecessor of p/q.
- $\omega = p + q$.
- τ is the tune of p/q, as in §2.6.
- κ is as in Lemma 4.1.

We mean to define ω, τ, κ for the other parameters as well. Thus, for instance, $\omega' = p' + q'$.

Lemma 5.1 *The following is true.*

1. *$\widehat{p}/\widehat{q} \in (0,1)$ is a rational in lowest terms.*
2. *Either $\tau - \tau' = \kappa\omega'$ or $\tau + \tau' = (1+\kappa)\omega'$. In all cases, $\tau' \leq \tau$.*
3. *p'/q' is also the even predecessor of $\widehat{p}/\widehat{q}$.*
4. *$\widehat{\kappa} = 0$.*
5. *$\omega - 2\tau = \widehat{\omega} - 2\widehat{\tau}$.*
6. *If $\kappa = 0$ then $\tau' = \tau$ when $\tau < \omega/4$ and $\tau' = \omega' - \tau$ when $\tau > \omega/4$.*
7. *If $\kappa \geq 1$ then $\kappa\widehat{\omega} < (3/2)\omega$.*

5.1.1 Statement 1

We first give a formula for p'/q'. There is some integer $\theta > 0$ so that

$$2p\tau = \theta(p+q) \pm 1.$$

Rearranging this, we get

$$p(2\tau - \theta) - q(\theta) = \pm 1$$

We have $\theta < \min(p, 2\tau)$. Setting

$$p'' = \theta, \qquad q'' = 2\tau - \theta,$$

we see that $|p''q - q''p| = 1$. This implies that p''/q'' is in lowest terms. Moreover, both p'' and q'' are odd. That means that $p' = p - p''$ and $q' = q - q''$. Hence

$$\omega' = \omega - 2\tau. \tag{43}$$

We have

$$\frac{\omega'}{\omega} = 1 - 2\Big(\frac{\tau}{\omega}\Big) \leq 1 - 2\Big(\frac{\kappa}{2\kappa+1}\Big) = \frac{1}{2\kappa+1}. \tag{44}$$

If $2\kappa q' \geq q$ then Equation 44 implies that $(2\kappa+1)p' + q' \leq p$. But then a short calculation shows that $|pq' - qp'| = p'q' > 1$. This is a contradiction. Hence $2\kappa q' < q$. The same argument works with p in place of q.

Since $\widehat{p}/\widehat{q}$ is Farey related to $p'/q' \in (0,1)$, we see that $\widehat{p}/\widehat{q}$ also lies in $(0,1)$ and is in lowest terms. This proves Statement 1.

5.1.2 Statement 2

We first prove that $\tau \equiv \pm\tau'$ mod ω'. We will assume that $2p\tau \equiv 1$ mod ω. The other case has a similar treatment. We use the formulas that we have just derived. We have

$$2p\tau = \theta\omega + 1 = \theta\omega' + 2\tau\theta + 1.$$

Rearranging, we get

$$\theta\omega' + 1 = 2(p - \theta)\tau = 2p'\tau.$$

Hence $2p'\tau \equiv \pm 2p'\tau' \bmod \omega'$. Since $2p'$ is relatively prime to ω', we get that $\tau \equiv \pm\tau' \bmod \omega'$.

Suppose first that $\tau \equiv \tau' \bmod \omega'$. The quantity $t = \tau - \kappa\omega'$ satisfies $2p't \equiv \pm 1 \bmod \omega'$. So, we just have to show that $t \in (0, \omega')$. We compute

$$t = \tau - \kappa\omega' = \tau - \kappa(\omega - 2\tau) = \tau(2\kappa + 1) - \kappa\omega. \tag{45}$$

After a bit of algebra, we see that this expression lies in $(0, \omega')$.

When $\tau \equiv -\tau \bmod \omega$, we instead consider the quantity $t = (\kappa + 1)\omega - \tau$. Again we know that $2p't \equiv \pm 1 \bmod \omega'$. The rest of the proof is very similar to what is done in the other case.

Now we establish the second part of Statement 1. We have one of the two equalities

$$\tau - \tau' = \kappa\omega', \qquad \tau + \tau' = (\kappa + 1)\omega'.$$

In the first case, we obviously have $\tau' \leq \tau$. In the second case, the same inequality follows from the fact that $\tau' < \omega'/2$.

5.1.3 Statement 3

Let $\widehat{p}/\widehat{q}$ be the core predecessor of p/q. Since $\widehat{p} = p - 2\kappa p'$ and $\widehat{q} = q - 2\kappa q'$ the two rationals p'/q' and $\widehat{p}/\widehat{q}$ are both Farey related. Equation 44, which is a strict inequality when $p > 1$, says that $(2\kappa+1)\omega' < \omega$. Since $\widehat{\omega} = \omega - 2\kappa\omega'$, we conclude that $\omega' < \widehat{\omega}$. Hence p'/q' is the Farey predecessor of $\widehat{p}/\widehat{q}$. This completes the proof of Statement 3.

5.1.4 Statement 4

If Statement 4 is false, then Equation 44, applied to $\widehat{p}/\widehat{q}$, tells us that

$$\widehat{\omega} - 3\omega' > 0. \tag{46}$$

Combining this with the fact that

$$\widehat{\omega} = \omega - 2\kappa\omega' > 0,$$

we can say that

$$\omega - (2\kappa + 3)\omega' > 0.$$

But then we have

$$\frac{1}{2\kappa + 3} = 1 - 2F_{\kappa+1} < 1 - 2\left(\frac{\tau}{\omega}\right) = \frac{\omega'}{\omega} < \frac{1}{2\kappa + 3}.$$

This is a contradiction.

5.1.5 Statement 5

The proof goes by induction on κ. The result is trivial for $\kappa = 0$, so we assume that $\kappa \geq 1$. We will change notation somewhat. Let $p_2/q_2 = p/q$ and let $p_1/q_1 = p'/q'$, the even predecessor of p_2/q_2. Define

$$p_3 = p_2 - 2p_1, \qquad q_3 = q_2 - 2q_1. \tag{47}$$

Then $\kappa_3 = \kappa_2 - 1$. It suffices to prove that $\omega_2 - 2\tau_2 = \omega_3 - 2\tau_3$. This is what we will do.

We have

$$w_3 = \omega_2 - 4\tau_2.$$

We define $u_3 = 3\tau_2 - w_2$. Note that $\omega_2 - 2\tau_2 = \omega_3 - 2u_3$. So, to finish the proof, we just have to check that $u_3 = \tau_3$. By hypothesis, we have $u_3 > 0$. Since $0 < \omega_2 - 2\tau_2 = \omega_3 - 2u_3$, we have $u_3 \in (0, \omega_3/2)$. So, we just have to check that $2p_3u_3 \equiv \pm 1 \mod \omega_3$. We will consider the case when $2p_2\tau_2 \equiv 1 \mod \omega_2$. The other case works the same way.

In the case at hand, we have

$$k_2 = \frac{2p_2\tau_2 - 1}{\omega_2}.$$

Using this equation, we compute

$$N := \frac{2p_3u_3 - 1}{\omega_3} = \frac{-3 - 2p_2^2 - 2p_2q_2 + 6p_2\tau_2}{\omega_2}.$$

Plugging in the equation $6p_2\tau_2 = 3 + 3k_2\omega_2$ and factoring, we see that N is an integer. Hence $2p_3u_3 \equiv 1 \mod \omega_3$.

5.1.6 Statement 6

When $\kappa = 0$, Statement 1 says that either $\tau = \tau'$ or $\tau + \tau' = \omega'$. Also, from Equation 43 we see that $2\omega' < \omega$ if and only if $\tau > \omega/4$.

Suppose that $\tau' = \tau$ and $\tau > \omega/4$. We have

$$\tau' = \tau > \omega/4 > \omega'/2.$$

which contradicts the fact that $\tau' < \omega'/2$. Suppose that $\tau' = \omega' - \tau$ and $\tau < \omega/4$. We have

$$2\tau' = 2\omega' - 2\tau > \omega - 2\tau = \omega,$$

which again is impossible.

5.1.7 Statement 7

We have

$$\widehat{\omega} = \omega - 2\kappa\omega' = \omega - 2\kappa(\omega - 2\tau) = (1-2\kappa)\omega + 4\kappa\tau.$$

Using this equality, we see that Statement 7 is equivaent to the truth of

$$\frac{\tau}{\omega} < \frac{4\kappa^2 - 2\kappa + 3}{8\kappa^2}. \tag{48}$$

By definition

$$\frac{\tau}{\omega} < \frac{\kappa+1}{2\kappa+3}.$$

Moreover,

$$\Big(\frac{4\kappa^2 - 2\kappa + 3}{8\kappa^2}\Big) - \Big(\frac{\kappa+1}{2\kappa+3}\Big) = \frac{9}{8\kappa^2(2+2\kappa)} > 0.$$

Therefore, Equation 48 holds.

5.2 Existence of the Predecessor Sequence

We begin by proving the existence of an auxiliary sequence. The notation $p'/q' \leftarrow p/q$ means that p'/q' is the even predecessor of p/q.

Lemma 5.2 *Let $A \in (0,1)$ be irrational. There exists a sequence $\{p_n/q_n\}$ converging to A such that $p_n/q_n \leftarrow p_{n+1}/q_{n+1}$ for all n.*

Proof: Let $\boldsymbol{H}^2$ denote the upper half-plane model of the hyperbolic plane. We have the usual Farey triangulation of $\boldsymbol{H}^2$ by ideal triangles. The geodesics bounding these triangles join rationals p_1/q_1 and p_2/q_2 such that $|p_1q_2 - p_2q_1| = 1$. We call these geodesics the *Farey Geodesics*. Two of the Farey geodesics in the tiling join the points $0/1$ and $1/1$ to $1/0$. This last point is interpreted as the point at infinity in the upper half plane model of $\boldsymbol{H}^2$. Let $Y = \{(x,y)|\ 0 < x < 1\}$ be the open portion of $\boldsymbol{H}^2$ between these two vertical geodesics.

Each Farey geodesic in Y which joins two even rationals p/q and p'/q' can be oriented so that it points from p/q to p'/q' if and only if $p'/q' \leftarrow p/q$. We leave the rest of the Farey geodesics unoriented.

We claim that there is a backwards oriented path from $0/1$ to A. To see this, choose any sequence of even rationals converging to A and consider their sequence of even predecessors. This gives us a sequence of finite directed paths joining $0/1$ to rationals which converge to A. Given the local finiteness of the Farey triangulation – meaning that any compact subset of $\boldsymbol{H}^2$ intersects only finitely many triangles – we can take a limit of these paths, at least on a subsequence, and get a directed path converging to A.

Reading off the vertices of this path gives us a sequence $\{p_n/q_n\}$ with $p_n/q_n \leftarrow p_{n+1}/q_{n+1}$ for all n. ♠

Proof of Lemma 4.1: We get the predecessor sequence from the sequence in Lemma 5.2 in the following way. Before each term p_n/q_n with $\kappa_n \geq 1$ we insert the rational $\widehat{p}_n/\widehat{q}_n$. This is the core predecessor of p_n/q_n. We know from the lemma above that $\widehat{\kappa}_n = 0$ and that p_{n-1}/q_{n-1} is the predecessor of $\widehat{p}_n/\widehat{q}_n$. In short

$$p_{n-1}/q_{n-1} \prec \widehat{p}_n/\widehat{q}_n \prec p_n/q_n.$$

Once we make all these insertions, the resulting sequence is a predecessor sequence which converges to A. Again, we will not stop to prove uniqueness because we don't care about it. ♠

5.3 Existence of the Approximating Sequence

We call the sequence from Lemma 5.2 the *even predecessor sequence*. As we have just remarked, the even predecessor sequence is a subsequence of the predecessor sequence.

Lemma 5.3 *Let $\{p_n/q_n\}$ be the even predecessor sequence which converges to A. There are infinitely many values of n such that $\tau_{n+1} > \omega_{n+1}/4$.*

Proof: If this lemma is false, then we might chop off the beginning and assume that this never happens. We have the formula $\omega_n = \omega_{n+1} - 2\tau_{n+1}$. So, if this lemma is false then we have $2\omega_n < \omega_{n+1}$ only finitely often. Chopping off the beginning of the sequence, we can assume that this never occurs.

We never have $2\omega_n = \omega_{n+1}$ because ω_{n+1} is odd. So, we always have $\omega_n > \omega_{n+1}$. In this case, we have

$$r_{n-1} = 2r_n - r_{n+1}$$

for $r \in \{p,q\}$. Applying this iteratively, we get that $r_k = (k+1)r_1 - kr_0$ for $k = 1,2,3,....$ But then $\lim p_n/q_n = (p_1-p_0)/(q_1-q_0)$, which is rational. This is a contradiction.

Now let p_n/q_n be a term with $2\omega_n < \omega_{n+1}$. We will consider the case when $p_n/q_n < p_{n+1}/q_{n+1}$. The other case has a similar treatment. We introduce the new rational

$$p_n^*/q_n^* = \frac{p_{n+1}-p_n}{q_{n+1}-q_n}.$$

Note that the three rationals p_n/q_n and p_{n+1}/q_{n+1} and p_n^*/q_n^* form the vertices of a Farey triangle. Moreover $p_{n+1}/q_{n+1} < p_n^*/q_n^*$ because p_{n+1}/q_{n+1} is the Farey sum of p_n/q_n and p_n^*/q_n^*.

Given the geometry of the Farey graph, we have

$$A \in [p_n/q_n, p_n^*/q_n^*].$$

Moreover

$$|p_n/q_n - p_n^*/q_n^*| = \frac{1}{q_n q_n^*} \leq \frac{1}{q_n^2} < \frac{4}{\omega_n^2}.$$

The second-to-last inequality comes from the fact that $q_n^* > q_n$. ♠

Proof of Lemma 4.2: Let $\{p_k/q_k\}$ be the even predecessor sequence. If it happens infinitely often that $\kappa_k \geq 1$ then there are infinitely many core terms in the predecessor sequence. Otherwise the predecessor sequence and the even predecessor sequence have the same tail end. But then the previous result shows that there are infinitely many values of k for which $2\omega_k < \omega_{k+1}$. This gives infinitely many strong terms. Suppose that $p/q, p^*/q^*$ are two consecutive terms in the predecessor sequence, with p/q non-weak.

Case 1: Suppose that p/q and p^*/q^* are both terms in the even sequence. Then $p/q \leftarrow p^*/q^*$ and $\tau_* > \omega_*/4$. But then $2\omega < \omega_*$, as in the proof of Lemma 5.3. From here, we get the Diophantine estimate from Lemma 5.3.

Case 2: Suppose that neither p/q not p^*/q^* are terms in the even sequence. This case cannot happen because when p^*/q^* is the core predecessor of some rational and p/q is the core predecessor of p^*/q^*. This contradicts Statement 4 of Lemma 5.1.

Case 3: Suppose that p/q is a term of the even sequence but p^*/q^* is not. Then the term in the predecessor sequence after p^*/q^* is p^{**}/q^{**}, which belongs to the even sequence. By Statement 3 of Lemma 5.1, we have $p/q \leftarrow p^{**}/q^{**}$. Moreover $2\omega < \omega^* < \omega^{**}$. From here, we get the Diophantine estimate from Lemma 5.3.

Case 4: Suppose that p/q is not a term in the even sequence but p^*/q^* is. Then p/q is the core predecessor of p^*/q^*. Statement 4 of Lemma 5.1 says that $\kappa = 0$. Hence the term p'/q' preceding p/q in the predecessor sequence is the even predecessor of p/q. Since $\kappa = 0$, we have $\tau < \omega/3$. Using the formula $\omega' = \omega - 2\tau$, we see that $3\omega' > \omega$. At the same time $\omega' < \omega < 2\omega^*$. Lemma 5.3 gives us

$$\left|A - \frac{p'}{q'}\right| < \frac{4}{(\omega')^2} < \frac{36}{\omega^2}.$$

At the same time

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = \frac{1}{qq'} < \frac{4}{\omega\omega'} < \frac{12}{\omega^2}.$$

The result in this case then follows from the triangle inequality. ♠

5.4 Another Identity

Let p/q be an even rational parameter with $\kappa \geq 1$. Let $\widehat{p}/\widehat{q}$ be the core predecessor. Define

$$w = \widehat{\tau}, \qquad h = \omega - 2\tau. \tag{49}$$

Here is an identity between these quantities.

Lemma 5.4

$$(2\kappa + 1)h + 2w = \omega. \tag{50}$$

Proof: Define

$$F_\kappa = \frac{\kappa}{2\kappa + 1}. \tag{51}$$

Let F be the union of all such rationals.

We rescale the first block so that it coincides with the unit square. This rescaling amounts to dividing by ω. For each quantity λ defined above, we let $\lambda^* = \lambda/\omega$. Within the unit square, our identity is

$$(2\kappa + 1)h^* + 2w^* = 1. \tag{52}$$

The rescaled horizontal and vertical lines depend continuously on the parameter τ^*. Moreover, when $\tau^* \in (F_\kappa, F_{\kappa+1})$ is rational, all the horizontal and vertical lines of capacity up to $4\kappa+2$ are distinct, because the denominator of the corresponding rational is at least $2\kappa+1$. Therefore, κ is a locally constant function of τ^*, and changes only when τ^* passes through a value of F.

Consider what happens when $\tau^* = F_\kappa + \epsilon$, and $\epsilon \in (0, F_{\kappa+1} - F_\kappa)$. We compute

$$(2\kappa+1)\tau^* - \kappa = (2\kappa+1)\epsilon < 1/2 \tag{53}$$

Hence $w = (2\kappa+1)\epsilon$, which means that $2w^* = (4\kappa+2)\epsilon$. We also have $h^* = 1 - 2\tau^*$. From these equations one can easily compute that Equation 52 holds for all $\epsilon \in (0, F_{\kappa+1} - F_\kappa)$. ♠

Lemma 5.5 $w = \widehat{\tau}$. *In other words, the line* $x = \widehat{\tau}$ *has capacity* $4\kappa+2$ *with respect to* p/q.

Proof: In view of Equation 50, we just have to prove that

$$(2\kappa+1)(\omega - 2\tau) + 2\widehat{\tau} = \omega \tag{54}$$

By Statement 5 of Lemma 5.1 we have $\omega - 2\tau = \widehat{\omega} - 2\widehat{\tau}$. But then Equation 54 is equivalent to

$$(2\kappa+1)\widehat{\omega} - 4\kappa\tau = \omega. \tag{55}$$

This equation holds, because

$$\omega - \widehat{\omega} = 2\kappa\omega' = 2\kappa(\omega - 2\tau) = 2\kappa(\widehat{\omega} - 2\widehat{\tau}) = 2\kappa\widehat{\omega} - 4\kappa\widehat{\tau}.$$

This completes the proof. ♠

6 The Box Lemma and The Copy Theorem

6.1 The Weak and Strong Copying Lemmas

Let p/q be an even rational parameter and let p'/q' be the even predecessor of p/q. Let Π and Π' denote the plaid tilings with respect to these two parameters. In this chapter we will consider the case when $\kappa = 0$. There are two subcases:

- **weak:** $2\omega' > \omega$. This corresponds to $\tau < \omega/4$.
- **strong:** $2\omega' < \omega$. This corresponds to $\tau > \omega/4$ (and $\tau < \omega/3$.)

Let $R_{p'/q'}$ be the rectangle associated to the parameter p'/q' in §4.2. Again, this rectangle is bounded by the lines

- $y = 0$.
- $y = \omega'$.
- $x = 0$.
- $x = \min(\tau', \omega - 2\tau')$.

The right side of $R'_{p'/q'}$ is whichever line of capacity at most 4 is closest to the y-axis.

In the weak case, we let Σ' denote the subset of of $R_{p'/q'}$ bounded above by the line

$$y = \omega' - \min(\tau', \omega' - 2\tau'). \tag{56}$$

That is, the top of Σ' is whichever horizontal line of capacity at most 4 lies closest to the top of $R_{p'/q}$.

In the strong case, we let $\Sigma' = R_{p'/q'}$.

In both cases we define

$$\Sigma = \Sigma'. \tag{57}$$

Even though $\Sigma = \Sigma'$ it is useful to have separate notation, so that in general an object X' corresponds to the parameter p'/q' and an object X corresponds to the parameter p/q.

In the next chapter we will prove the following result.

Lemma 6.1 (Weak and Strong Copying) $\Sigma' \cap \Pi' = \Sigma \cap \Pi$.

6.2 The Core Copy Lemma

Let p/q be an even rational parameter with $\kappa \geq 1$. Let $\widehat{p}/\widehat{q}$ be the core predecessor of p/q. Let Π and $\widehat{\Pi}$ denote the plaid tilings with respect to these two parameters.

Let Υ be vertical translation by $(\omega + \widehat{\omega})/2$. It follows from Statement 5 of Lemma 5.1 that Υ maps the horizontal lines of capacity ± 1 w.r.t $\widehat{p}/\widehat{q}$ to the lines of capacity ± 1 w.r.t. p/q.

Let $R_{\widehat{p}/\widehat{q}}$ is the rectangle associated to the parameter $\widehat{p}/\widehat{q}$ in §4.2. Define

$$\widehat{\Sigma} = R_{\widehat{p}/\widehat{q}}, \qquad \Sigma = \Upsilon(\widehat{\Sigma}). \tag{58}$$

In §9 we will prove the following result.

Lemma 6.2 (Core Copying) $\Sigma \cap \Pi = \Upsilon(\widehat{\Sigma} \cap \widehat{\Pi})$.

6.3 Proof of the Box Lemma

Our proof goes by induction on the denominator of the parameter. We will suppose that p/q is a rational with the smallest denominator for which we don't know the truth of the result.

By construction, Γ can only intersect ∂R in the right edge, which we call ρ. If we knew that Γ intersects ρ twice, then, because Γ is a closed loop, we could conclude that the portion of Γ contained in Σ is an arc whose endpoints are on ρ. If ρ has capacity 2, then Γ can intersect ρ at most twice, and the large x-diameter of Γ implie that Γ does intersect ρ. By symmetry, Γ intersects ρ twice.

Now suppose that ρ has capacity 4. In this situation, we have $\kappa \geq 1$, so we can apply the Core Copy Lemma to p/q and $\widehat{p}/\widehat{q}$.

Lemma 6.3 $\Sigma \subset R$.

Proof: The left edges of Σ and R both lie in the x-axis. So, we just have to show that the width of Σ is at most the width of R. The width of R is $\omega - 2\tau$. Statement 2 of Lemma 5.1 says that $\widehat{\kappa} = 0$. Hence, the width of $\widehat{\Sigma}$ is $\widehat{\tau}$, a quantity which not greater than $\widehat{\omega} - 2\widehat{\tau}$. (The latter quantity is the distance between the nearest line of capacity 4 to the x-axis.) Now, using Statement 5 of Lemma 5.1, we observe that $\widehat{\tau} \leq \widehat{\omega} - 2\widehat{\tau} = \omega - 2\tau$. ♠

Lemma 6.4 $\widehat{\Gamma}$ *does not intersect the right edge of* $\widehat{\Sigma}$.

Proof: Since $\widehat{\kappa} = 0$, the right edge of $\widehat{\Sigma}$ has capacity 2. $\widehat{\Gamma}$ only intersects the line containing this edge twice, and these intersection points must outside of $\widehat{\Sigma}$, for otherwise $\widehat{\Gamma}$ could not make a closed loop. Compare Figure 2.1. ♠

It now follows from the Core Copy Lemma that Γ does not intersect the right edge of Σ. Figure 6.1 shows three sub-rectangles $R_1, R_2, \Sigma \subset R$. The horizontal dividers in the picture are the horizontal lines of capacity 2. We have already shown that Γ does not intersect the right edge of Σ. But then Γ cannot cross ρ between the two horizontal lines of capacity 2. To finish our proof, we just have to check that Γ intersects the right edge of R_1 once. By symmetry, Γ intersects the right edge of R_2 once, and the right edge of Σ blocks Γ so that it cannot intersect ρ anywhere else.

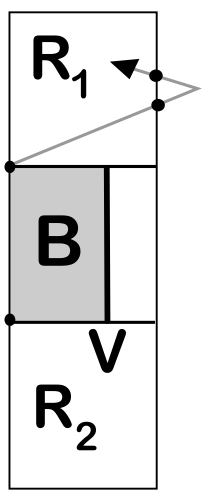

Figure 6.1: The rectangles R_1, S, R_2.

We know that Γ cannot intersect ρ_1 more than twice because then, by symmetry, Γ would intersect ρ at least 6 times. If Γ intersects ρ_1 exactly twice, then Γ is trapped in R_1 and cannot get around to close up with the portion of Γ outside R_1. Since Γ cannot get trapped in this way, we see that Γ intersects ρ_1 at most once. On the other hand, if Γ does not intersect ρ_1 at all, then Γ is trapped in the same way. In short, Γ intersects ρ_1 exactly once, and the same goes for ρ_2.

This completes the proof of the Box Lemma.

6.4 Setup for the Copying Theorem

Let $R_{p/q}$ be the rectangle associated to the parameter p/q.

Lemma 6.5 *Let p'/q' be the even predecessor of p/q. Suppose that $\kappa = 0$. Then $R' \subset R$.*

Proof: Both boxes contain $(0,0)$ as the bottom left vertex. So, we just have to show that the width and height of R' are at most the width and height of R. The width of R is τ. The width of R' is either τ' or $\omega' - 2\tau'$, whichever is smaller. In either case, we have $\text{width}(R') \leq \tau' \leq \tau = \text{width}(R)$. The height of R is ω and The height of R' is $\omega - 2\tau$. ♠

Suppose that p/q and p^*/q^* are two consecutive terms in the approximating sequence. We write

$$p/q = p_0/q_0 \prec p_1/q_1 \leftarrow \cdots \leftarrow p_n/q_n. \tag{59}$$

If p/q is strong then p_0/q_0 is the even predecessor of p_1/q_1, and p_k/q_k is weak for $k = 1, ..., n$. If p/q is core, then p_0/q_0 is the core predecessor of p_1/q_1 and p_1/q_1 is non-core, and p_k/q_k is weak for $k = 2, ..., n$. Here $k \geq 2$.

We introduce some notation to help with the proof. Let $(\Sigma'_k, \Sigma_{k+1})$ be the pair of rectangles associated to the pair of parameters $(p_k/q_k, p_{k+1}/q_{k+1})$. Note that p_k/q_k gets two such rectangles attached to it, namely Σ_k and Σ'_k. These rectangles play different roles in the proof. Let Π_k denote the plaid tiling associated to p_k/q_k. In general, we set $\omega_k = \omega(p_k/q_k)$, etc.

6.5 The Strong Case

Suppose first that p_0/q_0 is strong. Recalling that $\Sigma'_0 = \Sigma_1 = B_0$, we conclude that

$$B_0 \cap \Pi_0 = \Sigma'_0 \cap \Pi_0 =^* \Sigma_1 \cap \Pi_1 = B_0 \cap \Pi_1. \tag{60}$$

The starred equality comes from the Strong Copy Lemma.

Recall that TH_k and BH_k are the top and bottom horizontal lines of capacity 2 with respect to p_k/q_k.

Lemma 6.6 *For each $k = 1, ..., n$ the rectangle B_0 is contained in the lower half of B_k and one of BH_0 or TH_0 coincides with BH_k.*

Proof: Note that every box in sight contains $(0,0)$ as the lower left vertices. So, we can decide which box contains which other box just by looking at the widths and heights.

Eince $\tau_1 > 1/4$ we have $2\omega_0 < \omega_1$, the height of B_0 is less than half that of B_1. Hence B_0 lies in the lower half of B_1. By Lemma 6.5, we have $B_1 \subset ... \subset B_k$. Hence B_0 lies in the lower half of B_k.

From Statement 1 of Lemma 5.1, we have either $BH_0 = BH_1$ or $TH_0 = BH_1$. From Statement 6 of Lemma 5.1, we see that $BH_1 = ... = BH_k$. Hence either $BH_0 = BH_k$ or $TH_0 = BH_k$. ♠

Note that Σ'_k contains the lower half of B_k. Hence $B_0 \subset \Sigma'_k$ for all k. Now we will show inductively that $B_0 \cap \Pi_k$ implies $B_0 \cap \Pi_{k+1}$. We already have proved this for $k = 0$. For each $k = 1, ..., n-1$, we have $\tau_k < 1/4$. We have

$$B_0 \cap \Pi_0 = B_0 \cap \Pi_k \subset \Sigma'_k \cap \Pi_k =^* \Sigma_{k+1} \cap \Pi_{k+1}. \tag{61}$$

The starred equality comes from the Weak Copy Lemma. From this equation, we get $B_0 \cap \Pi_0 = B_0 \cap \Pi_{k+1}$. By induction, $B_0 \cap \Pi_0 = B_0 \cap \Pi_n$. That is, Π_n copies Π_0 inside B_0, and B_0 lies in the lower half of B_n. This is what we wanted to prove.

6.6 The Core Case

Suppose that p_0/q_0 is core. Let Υ be the vertical translation by $(\omega_1 - \omega_2)/2$. By construction $\Upsilon(B_0)$ is symmetric with respect to the horizontal midline of B_1. Statement 5 of Lemma 5.1 says that the distance between the two horizontal lines of capacity 2 is the same w.r.t p_0/q_0 and w.r.t. p_1/q_1. Hence, by symmetry, $\Upsilon(TH_0) = TH_1$ and $\Upsilon(BH_0) = BH_1$. We have

$$\Upsilon(B_0 \cap \Pi_0) = \Upsilon(\Sigma'_0 \cap \Pi_0) = \Sigma_1 \cap \Pi_1 = \Upsilon(B_0) \cap \Pi_1. \tag{62}$$

Lemma 6.7 $\Upsilon(B_0) \subset \Sigma'_1$.

Proof: The left edge of $\Upsilon(B_0)$ lies in the y-axis, just like the left edge of Σ'_1.

The width of $\Upsilon(B_0)$ is τ_0 and the width of Σ'_1 is $\omega_1 - 2\tau_1$. By Statement 2 of Lemma 5.1, the predecessor sequence cannot have 2 core terms in a row. Hence $\kappa_0 = 0$. This means that $3\tau_0 < \omega_0$. Hence

$$\tau_0 < \omega_0 - 2\tau_0 = \omega_1 - 2\tau_1.$$

The equality is Statement 5 of Lemma 5.1. This takes care of the widths.

The y-coordinate of the top edge of $\Upsilon(B_0)$ is $(\omega_0+\omega_1)/2$. The height of Σ_1' is either ω_1 or $2\tau_1$ depending on whether p_1/q_1 is strong or weak. Since $2\tau_1<\omega_1$, we just need to deal with the weak case. That is, we have to show that $\omega_0+\omega_1<4\tau_1$. But $4\omega_1/3<4\tau_1$ because p_0/q_0 is core. So, it suffices to show that

$$\omega_0<\omega_1/3.$$

But this follows from Equation 44, because $\kappa_1\geq 1$. ♠

Now we can take the next step in the argument.

$$\Upsilon(B_0\cap\Pi_0)=\Upsilon(B_0)\cap\Pi_1\subset\Sigma_1'\cap\Pi_1=^*\Sigma_2\cap\Pi_2. \tag{63}$$

The starred equality is either the Weak Copy Lemma or the Strong Copy Lemma, whichever applies. Hence $\Upsilon(B_0\cap\Pi_0)=\Upsilon(B_0)\cap\Pi_2$.

Lemma 6.8 *For each $k=2,...,n$ the rectangle $\Upsilon(B_0)$ is contained in the lower half of B_k and one of BH_0 or TH_0 coincides with BH_k.*

Proof: By Lemma 6.5, the widths of $R_1,...,R_k$ are non-decreasing. So, the width of $\Upsilon(B_0)$ is at most the width of R_k.

The y-coordinate of the top edge of $\Upsilon(B_0)$ is $(\omega_0+\omega_1)/2$. The height of B_2 is ω_2. By Statement 8 of Lemma 5.1, we see that the y coordinate of the top edge of $\Upsilon(B_0)$ is less than half the height of B_2. This takes care of the case $k=2$.

By Lemma 6.5, we have $B_2\subset...\subset B_k$, and all these boxes have $(0,0)$ as their bottom left vertex. Hence, $\Upsilon(B_0)$ lies in the bottom half of B_k.

We have already seen that $\Upsilon(BH_0)=BH_1$ and $\Upsilon(TH_0)=TH_1$. By Lemma 6.6, one of these two lines coincides with BH_2. But then Statement 6 of Lemma 5.1 says that $BH_2=...=BH_k$. ♠

The rest of the proof is the same inductive argument as in the Strong Case.

7 The Weak and Strong Copy Lemmas

7.1 Geometric Alignment

In this chapter we prove the Weak and Strong Copy Lemmas. The two results have essentially the same proof, except for a few minor details. The idea is to verify the conditions of the Matching Lemma from §2.5. We use the notation and terminology from §2.5. Here we have $\Upsilon =$ Identity and hence $\Sigma = \Sigma'$.

The two plaid tilings agree along the bottom edge of Σ and Σ', because this common edge lies in the boundary of the first block w.r.t. both parameters. Hence (Σ, Π) and (Σ', Π') are weakly horizontally aligned.

Lemma 7.1 (Σ, Π) *and* (Σ', Π') *are geometrically aligned.*

Proof: Let z and z' be corresponding points in $\Sigma = \Sigma'$. These points lie on slanting lines of the same type which have the same y-intercept. The difference in slopes of the two lines is

$$|P - P'| = |Q - Q'| = \frac{2}{\omega\omega'}.$$

Hence

$$\|z - z'\| \leq \frac{2\tau'}{\omega\omega'} < \frac{1}{\omega} < \frac{1}{\omega'} \tag{64}$$

But z' is at least $1/\omega'$ from the interval contaiing it. Hence z and z' lie in the same vertical unit interval. ♠

Remark: In replacing $1/\omega$ by $1/\omega'$, we threw away some of the strength of our estimate. We did this because in §9 we will have a much tighter estimate, and we want the two arguments to look similar.

7.2 Alignment of the Capacity Sequences

We introduce new variables

$$M_i = M_i\omega, \qquad C_j = c_j\omega, \qquad M'_i = m'_i\omega', \qquad C'_j = c'_i\omega'. \tag{65}$$

- C_i is a nonzero even integer in $(-\omega, \omega)$.
- C'_i is a nonzero even integer in $(-\omega', \omega')$.

- M'_j is an odd integer in $[-\omega', \omega']$.
- M_j is an odd integer in $[-\omega, \omega]$.

Let

$$\lambda = \omega'/\omega. \tag{66}$$

We have

$$C'_i = [2\omega' P' i]_{2\omega'}, \qquad C_i = [2\omega P i]_{2\omega}, \qquad \lambda C_i = [2\omega' P i]_{2\omega'}. \tag{67}$$

We introduce the expression

$$\Psi(i) = |2\omega P i - 2\omega' P' i|. \tag{68}$$

As long as

$$\Psi(i) < \ell + 2\lambda, \qquad \min(|C'_i|, \omega' - |C'_i|) \geq \ell \tag{69}$$

the signs of C'_i and C_i are the same.

Using the fact that

$$|P - P'| = \frac{2}{\omega\omega'} \tag{70}$$

we see that

$$\Psi(i) = \frac{4i}{\omega} \leq \frac{4W'}{\omega}. \tag{71}$$

In all cases, we have $W' \leq \tau' \leq \omega'/2$. Hence

$$\Psi(i) < 2\lambda < 1 + 2\lambda.$$

So, we can take $\ell = 1$ in Equation 69. Hence, the two capacity sequences are arithmetically aligned.

7.3 Alignment of the Mass Sequences

We have

$$M'_i = [\omega' P' i + \omega']_{2\omega'}, \qquad M_i = [2\omega P i + \omega]_{2\omega}, \qquad \lambda M_i = [\omega' P i + \omega']_{2\omega'}. \tag{72}$$

We introduce the expression

$$\Xi(i) = |\omega P i - \omega' P' i|. \tag{73}$$

This function differs from Ψ just in a factor of 2. As long as m'_i is a signed term and

$$\Xi(i) < \ell + \lambda, \qquad \min(|M'_i|, \omega' - |M'_i|) \geq \ell \tag{74}$$

the signs of M'_i and M_i are the same.

We have

$$\Xi(i) = \frac{2i}{\omega} < \frac{2H' + 4W'}{\omega}. \tag{75}$$

There are several cases to consider.

Case 1: Suppose that we are in the weak case and $\tau' \leq 2\omega' - \tau'$. We have

$$H' = \omega' - \tau', \qquad W' = \tau'.$$

Using the facts that

$$\tau' = \tau, \qquad \omega' = \omega - 2\tau,$$

and plugging the values of W' and H' into Equation 75, we get

$$\Xi(i) < \frac{2\omega' + 2\tau'}{\omega} = \frac{2\omega' + 2\tau}{\omega} = \frac{2\omega - 2\tau}{\omega} = \frac{\omega + \omega'}{\omega} = 1 + \lambda.$$

So, Equation 74 holds and there are no sign changes.

Case 2: Suppose that we are in the weak case and $\tau' > 2\omega' - \tau'$. This means that $\tau < \omega/4$ and $\tau' = \tau$ and $\tau' > \omega'/3$. We have

$$H' = 2\tau', \qquad W' = \omega - 2\tau'.$$

Plugging this into Equation 75, we get

$$\Xi(i) < \frac{4\omega' - 4\tau'}{\omega} < \frac{8\tau'}{\omega} = \frac{8\tau}{\omega} < 2.$$

So, we only have to worry about the case when $\widehat{C}_i = \pm 1$. This happens for $i \in \{-\tau', \tau', \omega' - \tau'\}$. The highest index is $\omega' + \tau' - 1$. we have

$$\Psi(\pm\tau') \leq \Psi(\omega' - \tau') = \frac{2\omega' - 2\tau'}{\omega} = \frac{2\omega' - 2\tau}{\omega} = \frac{2\omega - 4\tau}{\omega} =$$
$$\frac{\omega + (\omega - 4\tau)}{\omega} < \frac{\omega + \omega'}{\omega} = 1 + \lambda. \tag{76}$$

So Equation 74 holds in all cases.

Case 3: Suppose we are in the strong case and $\tau' \leq 2\omega' - \tau'$. We have

$$H' = \omega', \qquad W' = \tau'.$$

This gives us the estimate

$$\Xi(i) < \frac{2\omega' + 4\tau'}{\omega}. \tag{77}$$

Since $3\tau' \leq \omega'$ and $\tau + \tau' = \omega'$, we have

$$2\tau' \leq \omega' - \tau' = \tau.$$

Hence

$$\Xi(i) < \frac{2\omega' + 2\tau}{\omega} = \frac{(\omega' + 2\tau) + \omega'}{\omega} = 1 + \lambda.$$

So, Equation 74 holds in all cases, and there are no sign changes.

Case 4: Suppose we are in the strong case and $\omega' - 2\tau' < \tau'$. In this case, we have

$$H' = \omega', \qquad W' = \omega' - 2\tau'.$$

This gives us the estimate

$$\Xi(i) < \frac{6\omega' - 8\tau'}{\omega}. \tag{78}$$

Using the fact that $\tau + \tau' = \omega'$ and $\tau < \omega/3$, we get

$$\Xi(i) < \frac{6\tau - 2\tau'}{\omega} < 2.$$

Once again, we just have to worry about $C_i = \pm 1$. The relevant indices are $i \in \{-\tau', \tau', \omega' - \tau', \omega' + \tau'\}$. We have

$$\Psi(\pm\tau') \leq \Psi(\omega' - \tau') < \Psi(\omega' + \tau') = \frac{2\omega' + 2\tau'}{\omega} < \frac{\omega + \omega'}{\omega} = 1 + \lambda.$$

Here we used the fact that $2\omega' < \omega$ and $2\tau' < \omega'$. So, Equation 74 holds in all cases, and the mass sequences are aligned.

In short (Σ, Π) and (Σ', Π') are arithmetically aligned. We have verified all the conditions of the Matching Lemma, and so $\Sigma \cap \Pi = \Sigma' \cap \Pi'$. This proves the Weak and Strong Copy Lemmas.

8 The Core Copy Lemma

8.1 The Difficulty

We will again verify the criteria in the Matching Lemma from §2.5. We will change notation to reflect that we have been calling $\widehat{p}/\widehat{q}$ the core predecessor of p/q

The general idea of the proof here is similar to what we did in the previous chapter, but here we must work with a weaker estimate, namely

$$|P-\widehat{P}|=|Q-\widehat{Q}|=\frac{4\kappa}{\omega\widehat{\omega}}. \tag{79}$$

The poorer quality of the estimate in Equation 79 forces us to work harder in certain spots of the proof.

8.2 Weak Horizontal Alignment

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In our context here, recall that Υ is vertical translation by $\omega/2-\widehat{\omega}/2$.

Lemma 8.1 $\Upsilon(0,\widehat{\tau})=(0,\tau)$ *and* $\Upsilon(0,\widehat{\omega}-\widehat{\tau})=(0,\omega-\tau)$.

Proof: Since Υ is vertical translation to $(\omega-\widehat{\omega})/2$, we have

$$\Upsilon(\widehat{\tau})=\widehat{\tau}+(\omega-\widehat{\omega})/2=\tau.$$

The last equality is Statement 5 of Lemma 5.1. A similar calculation shows that $\Upsilon(\widehat{\omega}-\widehat{\tau})=\omega-\tau$. Here are have abused notation and just shown the action on the second coordinate. ♠

Lemma 8.2 (Σ,Π) *and* $(\widehat{\Sigma},\widehat{\Pi})$ *are weakly horizonally aligned.*

Proof: We take $\widehat{H}$ to be the line $y=\widehat{\tau}$ and we take H to be the line $y=\tau$. Since these lines have capacity 2 w.r.t. the relevant parameters, they only intersect the plaid tilings in the middle of the first interval. The second intersection points are outside $\widehat{\Sigma}$ and Σ respectively. ♠

8.3 Geometric Alignment

Lemma 8.3 (Σ,Π) *and* $(\widehat{\Sigma},\widehat{\Pi})$ *are geometrically aligned.*

Proof: Let $\widehat{z}$ and z be corresponding vertical intersection points. Let $\widehat{U}$ and U be the two vertical intervals respectively containing $\widehat{z}$ and z. We need to prove that $\Upsilon(\widehat{U})=U$.

Let $\widehat{L}$ and L be two slanting lines of the same type which contain $\widehat{z}$ and z respectively and have the same y-intercept. The two lines $\Upsilon(\widehat{L})$ and L have the same y intercept. The difference in their slopes is

$$\frac{4\kappa}{\omega\omega'}. \tag{80}$$

Hence

$$\|z-\Upsilon(\widehat{z})\| \leq \frac{4\kappa\widehat{\tau}}{\omega\widehat{\omega}} < \frac{2}{\widehat{\omega}}. \tag{81}$$

Here is a derivation of the final inequality. Statement 4 of Lemma 5.1 implies that $\widehat{\kappa}=0$. Hence $\widehat{\tau}<\widehat{\omega}/3$. But then

$$4\kappa\widehat{\tau} < \frac{4}{3}\kappa\widehat{\omega} < 2\omega.$$

The last inequality comes from Statement 7 of Lemma 5.1.

The estimate is not quite good enough. The problem is that $\widehat{z}$ is exactly $m/\widehat{\omega}$ from the endpoint of $\widehat{U}$, but it could happen that $m=1$. However, given the congruence properties of τ, and the slopes of the $\mathcal{P}$ and $\mathcal{Q}$ lines, we see that $m=1$ if and only if $\widehat{U}$ is contained in one of the two lines of capacity 2. If V is not one of these lines, we see that $\widehat{z}$ lies at least $2/\widehat{\omega}$ from the endpoints of $\widehat{U}$. Hence $\Upsilon(\widehat{z})$ lies at least $2/\widehat{\omega}$ from the endpoints of $\Upsilon(\widehat{U})$. But then our estimate tells us that $z\in\Upsilon(\widehat{U})$ as well. Hence $U=\Upsilon(\widehat{U})$.

Now we deal with the case when V is contains the right edges of Σ and $\widehat{\Sigma}$. The equation for V is $x=\widehat{\tau}$. Lemma 5.5 tells us that V has capacity $4\kappa+2$ w.r.t. p/q. This means that the y-coordinate of z has the form $(2\kappa+1)/\omega$ mod $\boldsymbol{Z}$. This will give us a good estimate provided that ω is large enough. To see that ω is large enough for a good estimate, note the formula

$$\omega=\omega'+2\kappa\widehat{\omega}.$$

Hence $\omega \geq 6\kappa + 3$, and z is at least $(2\kappa+1)/\omega$ from the endpoint of U. If $\Upsilon(\widehat{z})$ and z were contained in different vertical unit intervals, we would have

$$\frac{4\kappa\widehat{\tau}}{\omega\widehat{\omega}} \geq \frac{1}{\widehat{\omega}} + \frac{2\kappa+1}{\omega}.$$

This is impossible, because $\widehat{\tau}/\widehat{\omega} < 1/2$. (We don't even need the $1/\widehat{\omega}$ term on the right.) So, even in this special case, we see that $\Upsilon(\widehat{z})$ and z are in the same unit vertical interval. ♠

8.4 Alignment of the Capacity Sequences

Define

$$\lambda = \widehat{\omega}/\omega, \qquad \Psi(i) = |2\widehat{\omega}Pi - 2\widehat{\omega}\widehat{P}i|. \tag{82}$$

As long as Equation 69 holds, namely

$$\Psi(i) < \ell + 2\lambda, \qquad \min(|\widehat{c}_i|, \omega - |\widehat{c}_i|) \geq \ell$$

the signs of $\widehat{c}_i$ and C_i are the same.

Using Equation 79 and the fact that $i \in [1, \widehat{\tau}]$ we see that

$$\Psi(i) = \frac{8\kappa i}{\omega} \leq \frac{8\kappa\widehat{\tau}}{\omega}. \tag{83}$$

Statement 4 of Lemma 5.1 tells us that $\widehat{\kappa} = 0$, and this forces $\widehat{\tau} \leq \widehat{\omega}/3$. Combining this with Statement 7 of Lemma 5.1 we get $\Psi(i) < 4$. The only cases we need to worry about are

- $\widehat{C}_i = \pm 2$,
- $\widehat{C}_i = \pm(\widehat{\omega} - 1)$.
- $\widehat{C}_i = \pm(\widehat{\omega} - 3)$.

Case 1: We will consider the case when $\widehat{C}_i = 2$. The case when $\widehat{C}_i = -2$ has the same treatment. If $\widehat{C}_i = 2$ then $i = \widehat{\tau}$ and the corresponding $\mathcal{V}$ line is the right edge of $\widehat{\Sigma}$. But then from Lemma 5.5 we know that the capacity of this line w.r.t is $4\kappa + 2$. So, we either have $C_i = 4\kappa + 2$ as desired or $C_i = -(4\kappa+2)$. We will suppose that $C_i = -(4\kappa+2)$ and derive a contradiction. In this case, we would have

$$\Psi(i) = 2 + \lambda(4\kappa+2) = \frac{2\omega + 4\kappa\widehat{\omega} + 2\widehat{\omega}}{\omega}.$$

Combining this with Equation 83 we would get

$$2\omega + 4\kappa\widehat{\omega} + 2\widehat{\omega} < 8\kappa\widehat{\tau}.$$

Statement 4 of Lemma 5.1 says that $\widehat{\kappa} = 0$, which forces $\widehat{\tau} < \widehat{\omega}/3$, and this contradict the equation above.

Case 2: Lemma 2.5 tells us that the vertical lines of capacity $\pm(\widehat{\omega}-1)$ either occur at $x = \pm\widehat{\tau}/2$ or $x = \pm(\widehat{\omega}-\widehat{\tau})/2$, mod $\widehat{\omega}$. The latter case is irrelevant: The lines lie outside $\widehat{\Sigma}$. In the former case, the line of interest to is $x = \widehat{\tau}/2$. In other words $i = \widehat{\tau}/2$. Note that $2i = \widehat{\tau}$. The case $2i = \widehat{\tau}$ is the one just considered.

We will suppose that $\widehat{C}_{2i} = 2$. The case $\widehat{C}_{2i} = -2$ has the same treatment. When $\widehat{C}_{2i} = 2$, it means that

$$[\widehat{\omega}(2\widehat{P})(2i)]_{2\widehat{\omega}} = 2.$$

But this is the same as saying that

$$\widehat{\omega}(2\widehat{P})(2i) = 2\theta\widehat{\omega} + 2,$$

for some integer θ. But then

$$\widehat{\omega}(2\widehat{P}i) = 1 + \theta\widehat{\omega}.$$

Since the capacities are all even, this is only possible if θ is odd. But then

$$\widehat{C}_i = [\widehat{\omega}(2\widehat{P}i)]_{2\widehat{\omega}} = -\widehat{\omega} + 1.$$

So, $\widehat{C}_i$ and $\widehat{C}_{2i}$ have opposite signs. A similar argument shows that C_i and C_{2i} have opposite signs. Case 2 now follows from Case 1.

Case 3: When $\widehat{\tau}$ is even, the relevant line of capacity $\pm(\widehat{\omega}-3)$ is the line $x = 3\widehat{\tau}/2$. Since $3\widehat{\tau}/2 < \widehat{\omega}/2$, this line is the one which intesects the first block and is closer to the y-axis. But $3\widehat{\tau}/2 > \widehat{\tau}$, and so our line does not intersect $\widehat{\Sigma}$.

When $\widehat{\tau}$ is odd, the relevant line of capacity $\pm(\widehat{\omega}-3)$ is the line

$$x = \widehat{\omega}/2 - 3\widehat{\tau}/2.$$

It would happen that this line intersects $\widehat{\Sigma}$. We will deal with this line in a backhanded way.

We already know that $8\kappa\widehat{\tau}/\omega < 4$. If we really have a sign change in this case, we must have $\Psi(i) > 3$. But this means that

$$\frac{8\kappa\widehat{\tau}}{\omega} \in (3,4). \tag{84}$$

we have

$$\Psi(i) = \frac{8\kappa i}{\omega} = \frac{4\kappa\widehat{\omega}}{\omega} - \frac{12\kappa\widehat{\tau}}{\omega}. \tag{85}$$

By Statement 7 of Lemma 5.1, the first term on the right hand side of this equation is at most 6. By Equation 84, the second term on the right lies in $(9/2, 6)$. Therefore $\Psi(i) \leq 3/2$, and we have a contradiction. Hence, there are no sign changes, and the two capacity sequences are arithmetically aligned.

8.5 Calculating some of the Masses

As a prelude to checking that the mass sequences are arithmetically aligned, we take care of some special cases. This will make the general argument easier.

We define the mass and sign of an integer point on the y-axis to be the mass and sign of the $\mathcal{P}$ and $\mathcal{Q}$ lines containing them. Unless explicitly stated otherwise, these quantities are taken with respect to the parameter p/q. In this section we establish some technical results about some of the masses and signs.

Lemma 8.4 *Let $\widehat{p}/\widehat{q}$ be the core predecessor of p/q. Let Υ be the translation from the Core Copy Lemma.*

1. *Let Ω be the vertical interval of length $\widehat{\omega}$ and centered at the point $(0,\omega/2)$. The only points of mass less than $4\kappa+2$ contained in Ω are the points of mass 1.*

2. Υ *maps the points of mass* 1 *w.r.t* $\widehat{p}/\widehat{q}$ *to the points of mass* 1 *w.r.t.* p/q*, and in a sign-preserving way.*

3. Υ *maps the points of mass* $\widehat{\omega}-2$ *w.r.t* $\widehat{p}/\widehat{q}$ *to the points of mass* $\omega-2$ *w.r.t.* p/q*, and in a sign-preserving way.*

We will prove Lemma 8.4 through a series of smaller steps. We first dispense with a minor technical point.

Lemma 8.5 $3\widehat{\tau}<\widehat{\omega}$.

Proof: Since $\widehat{p}/\widehat{q}$ is assumed to be the nontrivial core predecessor of p/q, Statement 4 of Lemma 5.1 gives $\widehat{\kappa}=0$. This forces $3\widehat{\tau}\leq\widehat{\omega}$. The case of equality would force $\widehat{p}/\widehat{q}=1/2$. But then the even predecessor of p/q would be the even predecessor of $1/2$, by Statement 3 of Lemma 5.1. This is not possible. ♠

Proof of Statement 1: The points $(0,\tau)$ and $(0,\omega-\tau)$ are the two points of mass 1. We will give the proof when the sign of $(0,\tau)$ is positive. The other case is essentially the same.

Since Ω is symmetrically placed with respect to the horizontal midline of the first block $[0,\omega]^2$, it suffices to show that Ω does not contain any points of positive sign and mass $3,5,7,...,4\kappa+1$.

The endpoints of Ω are

$$(0,\omega/2-\widehat{\omega}/2), \qquad (0,\omega/2+\widehat{\omega}/2).$$

Our proof refers to the work in §5.4. In particular, $h=\omega-2\tau$ and $w=\widehat{\tau}$. Since the masses of the points on the Since the signed masses of the points on the y-axis occur in an arithmetic progression mod ω, the points on the y-axis having positive sign and mass $2\lambda+1$, at least for $\lambda=1,...,2\kappa$, are

$$(0,\tau-\lambda h)+(0,\omega\boldsymbol{Z}). \tag{86}$$

The second summand is included just so that we remember that the whole assignment of masses is invariant under translation by $(0,\omega)$. Also, we say "at least" because once λ is large enough the sign will change. So, it suffices to prove that

$$\tau-\lambda h\in[-\omega/2+\widehat{\omega}/2,\omega/2-\widehat{\omega}/2], \qquad \lambda=1,...,2\kappa.$$

Since these points occur in linear order, it suffices to prove the following two inequalities:

$$\tau - h < \omega/2 - \widehat{\omega}/2, \qquad \tau - 2\kappa h > -\omega/2 + \widehat{\omega}/2. \tag{87}$$

Since $h = \omega - 2\tau$, the first identity is equivalent to $3\omega - 6\tau > \widehat{\omega}$. But, by Statement 5 of Lemma 5.1,

$$3\omega - 6\tau = 3\widehat{\omega} - 6\widehat{\tau}.$$

But the inequality

$$3\widehat{\omega} - 6\widehat{\tau} > \widehat{\omega}.$$

is the same as the one proved in Lemma 8.5. This takes care of the first inequality.

It follows from Equation 50 in §5.4 that

$$\tau - 2\kappa h = -\omega + \tau + h + 2w.$$

Hence, the second inequality is the same as

$$(-\omega + \tau + h + 2w) - (-\omega/2 + \widehat{\omega}/2) > 0. \tag{88}$$

Plugging in $h = \omega - 2\tau$ and $w = \widehat{\tau}$, and using the relation $\omega - 2\tau = \widehat{\omega} - 2\widehat{\tau}$, we see that the left hand side of Equation 88 is just $\widehat{\tau}$. ♠

Remark: As a byproduct of our proof we note that the points $(0, \tau)$ and $(0, \tau + h + 2w)$ have the same sign. Also, $h + 2w = \widehat{\omega}$.

Proof of Statement 2: In view of Lemma 8.1, it suffices to prove that $(0, \widehat{\tau})$ has the same sign w.r.t. $\widehat{p}/\widehat{q}$ as $(0, \tau)$ has w.r.t. p/q. We will consider the case when $(0, \widehat{\tau})$ has positive sign w.r.t $\widehat{p}/\widehat{q}$. The opposite case has essentially the same treatment.

The argument from Case 1 from §8.4 tells us that the horizontal lines through $\widehat{y}_+$ have capacity $4\kappa + 2$ w.r.t. p/q. Since this is twice an odd integer, Lemma 2.5 tells us that the slanting lines through $\widehat{y}_+$ have mass $2\kappa + 1$. Now we observe the following.

$$P\omega' = P'\omega' + (P - P')\omega' = 2p' \pm \frac{2}{\omega} \equiv \frac{2}{\omega} \mod 2\boldsymbol{Z}. \tag{89}$$

The calculation in Equation 89 implies that the masses w.r.t. p/q along the y-axis (so to speak) change by ± 2 when the y-coordinate changes by ω'. Since

$$\tau = \widehat{\tau} + k\omega'$$

we see that the slanting lines through τ either have mass 1 or have mass

$$4\kappa + 1 \mod \omega.$$

But

$$\omega = \widehat{\omega} + 2\kappa\omega' > 6\kappa + 2.$$

Hence either the slanting lines through τ have mass 1 w.r.t p/q or they have mass $4\kappa + 1$. But we already know from Lemma 8.1 that these lines either have mass 1 or -1. Since $4\kappa + 1 \neq -1$ we know that the slanting lines through τ have mass 1, as desired. ♠

Proof of Statement 3: Let $z_\pm$ denote the points on the y-axis such that the slanting lines through these points have mass $\pm(\omega - 2)$. Likewise define $\widehat{z}_\pm$. We will treat the case when $(0, \tau)$ has positive sign. The other case has the same proof.

The function $f(x) \to [\widehat{P}x + 1]_2$ is locally affine, when the domain is interpreted to be the circle $\boldsymbol{R}/2\boldsymbol{Z}$. In the case at hand, this function changes by $+2$ when we move from $\widehat{\tau}$ to $\widehat{\omega} - \widehat{\tau}$. Given this property, and the fact that $f(0) = \pm\widehat{\omega}$, we see that

$$f(\widehat{\omega} - \widehat{2}\tau) = -\widehat{\omega} + 2.$$

But we also know that $y_+ = \tau$ and $y_- = \omega - \tau$ by the previous result. So, the same argument gives

$$f(\omega - 2\tau) = -\omega + 2.$$

This shows that $m_i = -\omega + 2$ when $\widehat{m}_i = -\widehat{\omega} + 2$. By the same argument, or symmetry, we see that $m_i = \omega - 2$ when $\widehat{m}_i = \widehat{\omega} - 2$. ♠

8.6 Alignment the Mass Sequences

We proceed as in §7.3. Let $\widehat{y} = \widehat{\omega}/2$ and $y = \omega/2$. Even though $\widehat{y}$ is an integer point, note that

$$[\widehat{P}\widehat{y} + 1]_2 = [\widehat{p} + 1]_2, \qquad [Py + 1]_2 = [p + 1]_2. \tag{90}$$

Since p and $\widehat{p}$ are integers having the same parity, the two expressions above are the same. Hence, there is an integer θ so that

$$\widehat{\omega}\widehat{P}\widehat{y} = \widehat{\omega}Py + \theta\widehat{\omega}.$$

Recall that $i_0 = (\omega - \widehat{\omega})/2$ and $y = \widehat{y} + i_0$. We can re-write the last equation as $\Psi(\widehat{\omega}/2) = 0$, where

$$\Xi(i) = |(\widehat{\omega}\widehat{P}i - \widehat{\omega}P(i + i_0) - \theta\widehat{\omega}| \tag{91}$$

The function Ξ has the same properties as the similar function from Equation 74: As long as

$$\Xi(i) < \ell + \lambda, \qquad \min(|\widehat{M_i}|, \widehat{\omega} - |\widehat{M_i}|) \geq \ell$$

the signs of $\widehat{M_i}$ and M_i are the same.

We say that the index i is *central* if $i \in (0, \widehat{\omega})$. Otherwise, we call i *peripheral*.

Lemma 8.6 *If i is a central index, and m_i' is term with a sign, then $\widehat{m}_i$ and m_i have the same sign.*

Proof: When i is a central index, we have $|i - \widehat{\omega}/2| \leq \widehat{\omega}/2$. Combining this with Equation 79, we get

$$\Psi(i) \leq \frac{4\kappa}{\widehat{\omega}\omega} \times \frac{\widehat{\omega}}{2} = \frac{2\kappa\widehat{\omega}}{\omega}. \tag{92}$$

Statement 7 of Lemma 5.1 says that $\kappa\widehat{\omega} < (3/2)\omega$. Combining this with the previous equation, we see that $\Psi(i) < 3$. This means that there are no sign changes unless $\widehat{m}_i = \pm 1$ or $\widehat{m}_i = \pm(\omega - 2)$. But Statements 2 and 3 of Lemma 8.4 take care of these special cases. ♠

Lemma 8.7 *If i is a peripheral index then $\widehat{m}_i$ and m_i have the same sign.*

Proof: Note that all the peripheral indices are signed terms, because none of these indices are in $\widehat{\omega}\boldsymbol{Z}$.

The index j of each peripheral term m_j differs from the index i of some central term by $\pm\widehat{\omega}$. We set

$$j = i + \widehat{\omega}. \tag{93}$$

The case when $j = i - \widehat{\omega}$ has a similar treatment.

Consider the situation w.r.t. the parameter $\widehat{p}/\widehat{q}$ first. These terms repeat every $\widehat{\omega}$. This meams that $m'_j = m'_i$.

Now observe that

$$[P\widehat{\omega}]_2 = [\widehat{P}\widehat{\omega} + (P - \widehat{P})\omega]_2 = [(P - \widehat{P})\omega]_2 = \pm\frac{4\kappa}{\omega}. \tag{94}$$

Therefore

$$m_j = m_i \pm 4\kappa. \tag{95}$$

By Statement 1 of Lemma 8.4, the central terms w.r.t. p/q either equal ± 1 or are greater than $4\kappa + 2$. So, if $m_i \neq \pm 1$, we have the terms m_i and m_j have the same sign. But then

$$\sigma(m_j) = \sigma(m_i) = \sigma(m'_i) = \sigma(m'_j).$$

Here σ denotes the sign. The middle equality comes from the previous result about the central indices.

What about when $m'_i = \pm 1$? In this case (since $j = i + \widehat{\omega}$) we must have $i = \tau$ and $j = \tau + \widehat{\omega}$. The remark following the proof of Statement 1 of Lemma 8.4. takes care of this case. ♠

Now we know that (Σ, Π) and $(\widehat{\Sigma}, \widehat{\Pi})$ are arithmetically aligned. We have verified the conditions of the Matching Lemma, and so $\Sigma \cap \Pi = \Upsilon(\widehat{\Sigma} \cap \widehat{\Pi})$. This completes the proof of the Core Copy Lemma.

9 References

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