Generalizing Gauss's Pentagon Relation

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1 Introduction

Gauss's pentagon relation says that the map

$$G_5(x_1,x_2)=\Big(x_2,\frac{1-x_1}{1-x_1x_2}\Big) \tag{1}$$

has order 5. That is, $G_5^5=I$, where I is the identity map. This fact is closely related to *Napier's rule* from spherical geometry, and it is also related to the projective geometry of pentagons. See [**MOT**], and also the discussion below.

Though it seems trivial, we point out that $G_5^{10}=I$ as well. In this paper we will show that G_5 is the first map in a series of maps $G_n:\boldsymbol{R}^{2n-8}\to\boldsymbol{R}^{2n-8}$, for $n=5,6,7,...$, such that $G_n^{2n}=I$. The next two maps are

$$G_6(x_1,x_2,x_3,x_4)=\Big(x_2,x_3,x_4,\frac{1-x_1-x_3+x_1x_2x_3}{1-x_1-x_3x_4}\Big) \tag{2}$$

$$G_7(x_1,x_2,x_3,x_4,x_5,x_6)=$$
$$\Big(x_2,x_3,x_4,x_5,x_6,\frac{1-x_1-x_3-x_5+x_1x_2x_3+x_3x_4x_5+x_1x_5}{1-x_1-x_3+x_1x_2x_3+x_1x_5x_6-x_5x_6}\Big) \tag{3}$$

The reader who has a computer handy can check that $G_6^{12}=I$ and $G_7^{14}=I$ on random inputs. The general equation has the form

$$G_n(x_1,...,x_{2n-8})=\Big(x_2,...,x_{2n-8},R_n(x_1,...,x_{2n-8})\Big), \tag{4}$$

where R_n is the rational function given in §23.

I discovered these maps when considering the projective geometry of polygons, in connection with the pentagram map. The pentagram map has deep connections to integrable systems and cluster algebras – see the references– and perhaps there is some depth to the maps discussed here.

2 Flag Invariants

We work in the projective plane. What we say works (with suitable interpretations) over any field, though for concreteness we think of the action taking place over $\boldsymbol{R}$.

We list points in homogeneous coordinates. LL' denotes the intersection of lines L and L', and PP' denotes the line containing P and P'. We have the *inverse cross ratio*

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-c)(b-d)}. \tag{5}$$

For the reader who wants to make their own experiments, we mention an extremely useful calculation device. Given 4 collinear points A, B, C, D in the projective plane, the quantity

$$\frac{(A \times B) * (C \times D)}{(A \times C) * (B \times D)} \tag{6}$$

is a vector of the form (x, x, x), where x is the inverse cross ratio. Here $(*)$ denotes pointwise multiplication, and the division line denotes pointwise division.

A *polygonal ray* is an infinite collection of points $P_{-7}, P_{-3}, P_{+1}, ...$, with indices congruent to 1 mod 4, normalized so that

$$P_{-7} = (0, 0, 1), \qquad P_{-3} = (1, 0, 1), \qquad P_{+1} = (1, 1, 1), \qquad P_{+5} = (0, 1, 1). \tag{7}$$

These points determine lines

$$L_{-5+k} = P_{-7+k}P_{-3+k}, \tag{8}$$

We define

$$F_{-6+k} = (P_{-7+k}, L_{-5+k}), \qquad F_{-4+k} = (P_{-3+k}, L_{-5+k}) \tag{9}$$

$$c(F_{0+k}) = [P_{-7+k}, P_{-3+k}, L_{-5+k}L_{3+k}, L_{-5+k}L_{7+k}], \tag{10}$$

$$c(F_{2+k}) = [P_{9+k}, P_{5+k}, L_{7+k}L_{-1+k}, L_{7+k}L_{-5+k}], \tag{11}$$

All these equations are meant for $k = 0, 4, 8, 12, ...$ Finally, we define

$$x_k = c(F_{2k}); \qquad k = 0, 1, 2, 3... \tag{12}$$

We have associated the *flag invariants* $x_0, x_1, x_2, ...$ to the polygonal ray. See [**Sch3**] for more details.

3 The Reconstruction Formula

Given a list $(x_0, x_1, x_2, ...)$, we seek a polygonal ray which has this list as its flag invariants.

We repeat some definitions from [**Sch3**]. An *odd block* is a sequence either of the form k or $k, k+1, k+2$, where k is odd. We say that two blocks are *well separated* if there are at least 3 integers strictly between them. For instance 1 and $3, 4, 5$ are not well separated, but 1 and $5, 6, 7$ are well separated.

We say that an *admissible sequence* is a finite increasing sequence of integers that decomposes into pairwise well-separated blocks. We define the *sign* of an admissible sequence to be $(+)$ if there are an even number of singles and $(-)$ if there are an odd number of singles. The emptyset counts as an admissible sequence, and its sign is $(+)$.

We attach a monomial to each admissible sequence I, as follows.

$$m(I) = \text{sign}(I) x^I. \tag{13}$$

For instance, when $I = (1, 5, 6, 7, 11)$ we have $m(I) = +x_1 x_5 x_6 x_7 x_{11}$.

Given odd integers $a < b$ we define S_a^b to be the set of admissible sequences which only use integers in the open interval (a, b). So, a and b themselves cannot appear in members of S_a^b. Finally, we define

$$O_a^b = \sum_{I \in S_a^b} m(I). \tag{14}$$

These polynomials are some of the building blocks for the integrable structure of the pentagram map; see [**Sch3**], [**OST1**], [**OST2**], and [**Sol**].

For the purposes of giving a recursive definition of these polynomials, we define $O_{-1}^{-1} = 1$ and $O_{b+2}^b = 0$. We then have

$$O_b^b = O_{b-2}^b = 1, \qquad O_a^b = \begin{pmatrix} 1 \\ -x_{b-2} \\ x_{b-4} x_{b-3} x_{b-2} \end{pmatrix} \cdot \begin{pmatrix} O_a^{b-2} \\ O_a^{b-4} \\ O_a^{b-6} \end{pmatrix}, \qquad a = b-4, b-6, ... \tag{15}$$

Taking [**Sch3**, eq. 20] and applying a suitable projective duality, we get the *reconstruction formula*

$$P_{9+2k} = \begin{pmatrix} 1 & -1 & x_0 x_1 \\ 1 & 0 & 0 \\ 1 & 0 & x_0 x_1 \end{pmatrix} \begin{pmatrix} O_{-1}^{3+k} \\ O_{+1}^{3+k} \\ O_{+3}^{3+k} \end{pmatrix}, \qquad k = 0, 2, 4, ... \tag{16}$$

4 Closed Polygons

Multiplying through by the matrix M^{-1}, where M is in the matrix in Equation 16, we get an alternate normalization. Setting

$$Q_{-7} = \begin{pmatrix} 0 \\ x_0x_1 \\ 1 \end{pmatrix}, \quad Q_{-3} = \begin{pmatrix} 0 \\ 0 \\ x_0x_1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{17}$$

we have

$$Q_{9+2k} = \begin{pmatrix} O_{-1}^{3+k} \\ O_{+1}^{3+k} \\ O_{+3}^{3+k} \end{pmatrix}, \qquad k = 0, 2, 4, ... \tag{18}$$

In case we have a closed n-gon, we have

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [Q_{-3}] = [Q_{4n-3}] = [Q_{9+2(2n-6)}] = \begin{bmatrix} O_{-1}^{2n-3} \\ O_{+1}^{2n-3} \\ O_{+3}^{2n-3} \end{bmatrix}. \tag{19}$$

Here $[\cdot]$ denotes the equivalence class in the projective plane. Equation 19 yields $O_{-1}^{2n-3} = O_{+1}^{2n-3} = 0$. Shifting the vertex labels of our polygon by 1 unit has the effect of shifting the flag invariants by 2 units. Doing all cyclic shifts, we get

$$O_a^b = 0 \qquad b - a = 2n-4, 2n-2, \qquad a, b \text{ odd.} \tag{20}$$

This last equation is consistent with Equation 17.

We can get even nicer relations if we consider projective duality. Given a polygon P with flag invariants $x_1, x_2, ...$ we consider the dual polygon P^*. The polygon P^* is such that a projective duality carries the lines extending the edges of P^* to the points of P, and *vice versa*. When suitable labeled, the flag invariants of P^* are $x_2, x_3, ...$ Equation 20 also holds for P^*.

We defined O_a^b with respect to the odd parity. We can make the same definition with respect to pairs $a < b$ when both have even parity. Since Equation 20 also holds relative to P^*, we do not need to take a and b odd in Equation 20. In particular, we have the $2n$ relations

$$O_a^b = 0 \qquad b - a = 2n - 4 \tag{21}$$

The fact that we can reconstruct a polygon from these relations alone can be interpreted as a proof that the relations in Equation 21 imply the relations in Equation 20.

5 The Rational Maps

Equation 21 tells us that $O_{-1}^{2n-5} = 0$. But now Equation 15 gives

$$x_{2n-7}x_{2n-8}x_{2n-9}O_{-1}^{2n-11} - x_{2n-7}O_{-1}^{2n-9} + O_{-1}^{2n-7} = 0. \tag{22}$$

note that x_0 does not occur in this equation. Solving for x_{2n-7}, we get

$$x_{2n-7} = R_n(x_1, ..., x_{2n-8}),$$

where

$$R_n(x_1, ..., x_{2n-8}) = \frac{O_{-1}^{2n-7}}{O_{-1}^{2n-9} - x_{2n-8}x_{2n-9}O_{-1}^{2n-11}} \tag{23}$$

Thanks to Equation 21, these equations hold when we shift the indices cyclically by any amount. Thus

$$x_{2n-7+k} = R_n(x_{k+1}, ..., x_{2n-8+k}), \qquad k = 1, ..., 2n. \tag{24}$$

In particular, if we define G_n as in Equation 4, relative to R_n from Equation 23, then we have $G_n^{2n} = I$.

Just to see that Gauss's relation comes out of this, consider the case $n = 5$. In this case

$$R_5(x_1, x_2) = \frac{O_{-1}^3}{O_{-1}^1 - x_2x_1O_{-1}^{-1}} = \frac{1 - x_1}{1 - x_1x_2} \tag{25}$$

The case $n = 6$ yields

$$R_6(x_1, x_2, x_3, x_4) = \frac{O_{-1}^5}{O_{-1}^3 - x_4x_3O_{-1}^1} = \frac{1 - x_1 - x_3 + x_1x_2x_3}{1 - x_1 - x_3x_4} \tag{26}$$

This agrees with what we advertised in the introduction.

One might wonder why $G_n^n = I$ for $n = 5$ but not for higher values of n. It turns out that the flag invariants for a pentagon satisfy $x_{k+5} = x_k$ for all k. See [**FT**] for a proof. This explains why, in fact, $G_5^5 = I$. In general, there is no such symmetry, and we have to double the period to get the identity. The special points on which R_n^n is the identity correspond to polygons with 2-fold projective symmetry when n is even, and certain self-dual polygons when n is odd. See [**FT**] for a detailed discussion of self-dual polygons.

6 References

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