Recurrence of the Pentagram Map

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1 Introduction

Given a convex n-gon, P, one can connect every other vertex with a line segment, creating a star-like figure called the pentagram. Part of the pentagram defines a new convex n-gon, P', as shown (for $n = 7$) in the first part of Figure 1. Iterating, one defines P'', P''', etc. The second part of Figure 1 suggests that the map $P \to P''$ naturally defines a map between labelled n-gons. We call the map $P \to P''$ the *pentagram map*. We studied the pentagram map in [**S**].

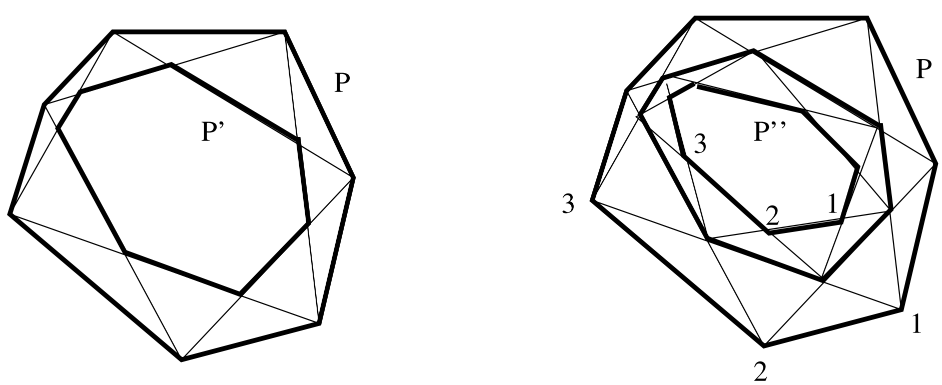

Figure 1

The natural setting for the pentagram map is the projective plane, $\boldsymbol{RP}^2$. (See §2 for background information on the projective plane.) Say that two

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labelled strictly convex n-gons are *equivalent* if there is a projective transformation of $\boldsymbol{RP}^2$ which takes one to the other. Let Σ_n denote the space of equivalence classes of strictly convex n-gons. As we saw in [**S**], the space Σ_n is diffeomorphic to $\boldsymbol{R}^{2n-8}$.

The pentagram map commutes with real projective transformations, and so induces a mapping $T_n : \Sigma_n \to \Sigma_n$, for all $n \geq 5$. We saw in [**S**] that the maps T_5 and T_6^2 act trivially on Σ_5 and Σ_6 respectively. For $n \geq 7$, the map T_n does not have finite order. In this paper we verify [**S**, Conjecture 4.1]:

Theorem 1.1 *T_n is recurrent on Σ_n, for all $n \geq 5$.*

By *recurrent*, we mean that almost every point is an accumulation point of its own forward orbit. Our result has the following geometric interpretation. Begin with a random choice of convex polygon P, and look at the sequence $P, T_n(P), T_n^2(P), \ldots$ From varying perspectives, one sees a near copy of P appear infinitely often.

Here is an outline of our proof of Theorem 1.1. In [**S**] we constructed a smooth function $f : \Sigma_n \to [1, \infty)$ with the following properties:

1. $f^{-1}[1, r]$ is compact for any real number $r > 1$.
2. $f \circ T_n = f$.

To help keep this paper self-contained, and also to set up some notation needed for later steps, we will construct f in a new way and sketch proofs of the two properties above. This is done in §3.

The two properties above show that $f^{-1}[1, r]$ is a compact T_n-invariant set. The main step in the proof of Theorem 1.1 is

Lemma 1.2 (Volume Lemma) *There exists a smooth volume form μ_n on Σ_n which is preserved by T_n.*

We will prove the Volume Lemma in §4.

By Sard's theorem, almost every choice of r yields a smooth manifold-with-boundary $X = f^{-1}[1, r]$. The map $T = T_n$ acts as a volume preserving map on X. To deal with this situation we invoke a special case of the Poincare Recurrence Lemma. The proof is short, we include it here. See [**A**] for more details.

Lemma 1.3 (Poincare Recurrence) *Suppose that $X \subset \boldsymbol{R}^m$ is a smooth compact manifold-with-boundary. Suppose $T : X \to X$ preserves a smooth volume form, defined in a neighborhood of X. Then almost every point $x \in X$ is an accumulation point of the sequence $\{T^k(x)|\ k \in \boldsymbol{N}\}$.*

Proof: For any $\epsilon > 0$ let N_ϵ be the Borel set of points $x \in X$ such that the sequence $\{T^k(x)|\ k \in \boldsymbol{N}\}$ avoids the ϵ-neighborhood of x. If N_ϵ has positive measure then we can find a δ-ball $B \subset X$, such that $\beta = B \cap N_\epsilon$ has positive measure. Here we take $\delta < \epsilon$. Since X has finite volume, the sets in $\{T^k(\beta)\}$ are not all pairwise disjoint. Hence, $T^i(\beta) \cap T^j(\beta)$ for some pair $i, j \in \boldsymbol{N}$. Setting $k = j - i$, we have $T^k(\beta) \cap \beta \neq \emptyset$. This contradiction shows that N_ϵ has measure zero. Since ϵ is arbitrary, we are done. ♠

Applying the Poincare Recurrence Lemma, we see that T_n is recurrent on $f^{-1}[1, r]$ for almost every choice of real $r > 1$. We choose a sequence $r_1, r_2, \ldots$ which increases unboundedly, such that T_n is recurrent on $f^{-1}[1, r_k]$ for all k. Since these sets exhaust Σ_n, we see that T_n is recurrent on Σ_n.

The recurrence property is more general than our result suggests. Let Ω_n be the set of projective equivalence classes of n-gons. These n-gons need not be convex. T_n is defined on a full measure set of Ω_n. As we conjectured in [**S**], it seems that $T_n : \Omega_n \to \Omega_n$ is also recurrent. Our proof here has nothing to say about this.

This paper relies on some basic projective geometry. In §2 we give some background information on this subject. More information on projective geometry can be found in [**H**], for instance.

All the ideas for our proof, save one, came from computer experimentation. In particular, we discovered all the computations in the paper numerically. On the negative side, some of our computations are unmotivated. We don't really understand why they are true. On the positive side, we know for sure that they *are* true. For instance, the main thrust in this paper is that a certain collection of matrices always has determinant 1. We computed this determinant on millions of random samples from this family and numerically it was as close to 1 as one could expect from a finite precision calculation.

A key idea in this paper, which did not come from computer experimentation, is the notion of the corner invariant f_p, defined in [**S**] and recalled here in §3.1. We originally learned about f_p from John Conway.

2 Projective Geometry

2.1 The Projective Plane

The real projective plane, $\boldsymbol{RP}^2$, is the space of one dimensional subspaces of $\boldsymbol{R}^3$. The ordinary plane, $\boldsymbol{R}^2$, can be considered as a subset of $\boldsymbol{RP}^2$ in the following way: One identifies the linear subspace spanned by the vector $(x, y, 1)$ with the point $(x, y) \in \boldsymbol{R}^2$. Under this embedding, $\boldsymbol{RP}^2$ is a compactification of $\boldsymbol{R}^2$. It is not hard to see that $\boldsymbol{RP}^2$ naturally has the structure of a smooth manifold.

A *line* in $\boldsymbol{RP}^2$ is the union of all 1-dimensional subspaces contained in a given 2-dimensional subspace. Lines in $\boldsymbol{RP}^2$ are actually topologically equivalent to circles. The set $\boldsymbol{RP}^2 - \boldsymbol{R}^2$ is exactly a line in $\boldsymbol{RP}^2$, known as the *line at infinity*. Every ordinary line in $\boldsymbol{R}^2$ extends to a line in $\boldsymbol{RP}^2$ by adding in the point where it intersects the line at infinity.

The lines and points in $\boldsymbol{RP}^2$ are intimately related. Given any two distinct points in $\boldsymbol{RP}^2$ there is a unique line which contains both of them. Likewise, given any two distinct lines in $\boldsymbol{RP}^2$ there is unique point contained on both.

2.2 Projective Transformations

An invertible linear transformation of $\boldsymbol{R}^3$ maps one dimensional subspaces to one dimensional subspaces. Thus, every such linear transformation induces a diffeomorphism of $\boldsymbol{RP}^2$. This diffeomorphism is called a *projective transformation*. Projective transformations act in such a way as to map lines in $\boldsymbol{RP}^2$ to lines in $\boldsymbol{RP}^2$.

The group of projective transformations is usually denoted by $PGL_3(\boldsymbol{R})$. It is an 8-dimensional Lie group. We say that a collection of points in $\boldsymbol{RP}^2$ is in *general position* if no three are contained in the same line. Say that a *quadrilateral* is a collection of 4 general position points in $\boldsymbol{RP}^2$. Each element of $PGL_3(\boldsymbol{R})$ is determined by its action on a quadrilateral. Indeed, given two quadrilaterals, with points labelled, there is a unique element of $PGL_3(\boldsymbol{R})$ which maps one quadrilateral to the other.

2.3 The Cross Ratio

Suppose that p_1, p_2, p_3, p_4 are 4 points on a line $L \subset \boldsymbol{RP}^2$. One defines the *cross ratio* $\chi(p_1, p_2, p_3, p_4)$ in the following way. First, use an element of $PGL_3(\boldsymbol{R})$ to identify L with (the one point extension of) the x-axis in $\boldsymbol{R}^2$. Let x_j be the image of p_j under this identification. Then define

$$\chi(p_1, p_2, p_3, p_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}$$

This definition is independent of any choices used in identifying L with the x-axis. $\chi(p_1, p_2, p_3, p_4)$ is invariant under projective transformations. That is

$$\chi(T(p_1), T(p_2), T(p_3), T(p_4)) = \chi(p_1, p_2, p_3, p_4); \qquad T \in PGL_3(\boldsymbol{R}).$$

2.4 The Hilbert Metric

We say that a set $X \subset \boldsymbol{RP}^2$ is *convex* if there is a projective transformation T such that $T(X)$ is a closed compact convex subset of $\boldsymbol{R}^2$.

If $X \subset \boldsymbol{RP}^2$ is a convex set, we can define a canonical metric d_X on its interior X^o. Given unequal points $p_2, p_3 \in X^o$, let L be the line containing p_2 and p_3. Let p_1 and p_4 be the two points where L intersects the boundary ∂X. We order these points so that p_1, p_2, p_3, p_4 come in order on L. We define

$$d_X(p_2, p_3) = \log \chi(p_1, p_2, p_3, p_4).$$

Note that $d(p_2, p_3) \to 0$ as $p_2 \to p_3$ and that $d(p_2, p_3) = d(p_3, p_2)$. The triangle inequality is also not hard to verify. d_X is known as the *Hilbert metric*.

Defined as it is, in terms of cross ratios, the Hilbert metric is natural with respect to $PGL_3(\boldsymbol{R})$. If X and Y are convex sets and $T : X \to Y$ is a projective transformation mapping X to Y then T is an isometry as measured relative to the two Hilbert metrics.

3 The Invariant Function

3.1 Basic Definition

Let P be an n-gon. We give an orientation to P, and we represent this orientation by an arrow, as shown in Figure 3.1. Let p be a vertex of P and let $a, b, ..., h, i$ be the points shown in Figure 3.1. We define

$$O_p(P) = \chi(a, b, c, d); \qquad E_p(P) = \chi(d, e, f, g); \qquad f_p(P) = \chi(b, h, i, f).$$

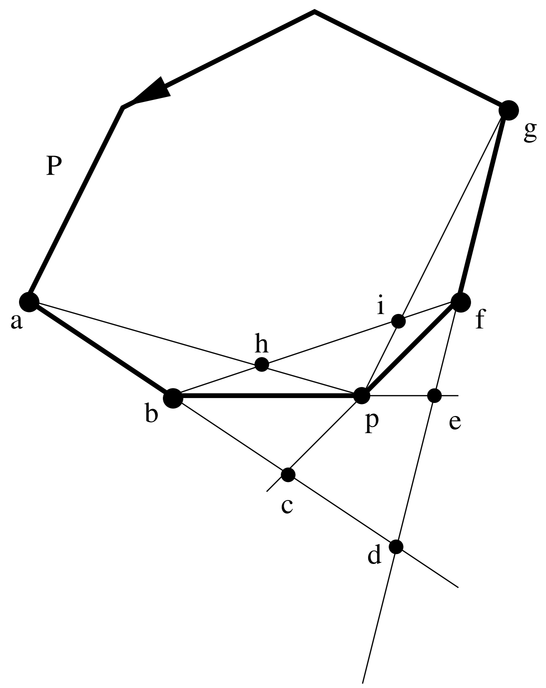

Figure 3.1

The quantity $f_p(P)$ is what we called the *corner invariant of* P *at* p in [**S**]. We define

$$f(P) = \prod_p f_p(P); \qquad O(P) = \prod O_p(P); \qquad E(P) = \prod E_p(P).$$

The product is taken over all vertices of P.

A short calculation shows that

$$f_p(P) = O_p(P)E_p(P).$$

To simplify the calculation, one can use the projective in variance to normalize so that the 4 vertices a, b, f, g form a unit square. We omit the details.

Taking the product of this identity over all vertices, we see that

$$f(P) = O(P)E(P).$$

Remark: Here is a geometric interpretation of $f(P)$. Let P' be the pentagram of P. Let X be the convex subset of $\boldsymbol{RP}^2$ whose boundary is P. Let d_X be the Hilbert metric on X. Let $p_1', ..., p_n'$ be the vertices of P' listed in order. By simply using the definition of the Hilbert metric, we have

$$\log f(P) = \sum_{i=1}^{n} d_X(p_i', p_{i+1}').$$

Here indices are taken mod n. In other words, $\log f(P)$ is the perimeter of P' as measured in the Hilbert metric on the convex set bounded by P. This interpretation shows that $f(P) > 1$.

3.2 Compactness Proof

In this section we prove that the level sets $f^{-1}[1, r] \subset \Sigma_n$ are compact. Our argument here is pretty much a repeat of what we said in [**S**]. Given an n-gon P, the corner invariants $f_p(P)$ all lie in $[1, \infty)$. Thus, if $f(P) \in [1, r]$ then $f_p(P) \in [1, r]$ for every vertex p of P.

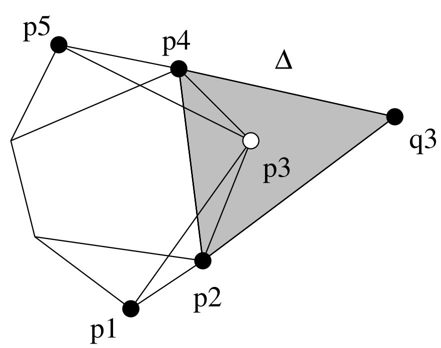

Figure 3.2.

Let p_1, p_2, p_3, p_4, p_5 be 5 consecutive vertices of P, as shown in Figure 3.2. The point p_3 is confined to the shaded open triangle Δ whose vertices are p_2, p_4 and q_3. Here

$$q_3 = \overline{p_1 p_2} \cap \overline{p_4 p_5}.$$

We set $f_j = f_{p_j}(P)$. Suppose that p_1, p_2, p_4, p_5 are held fixed and that p_3 (and possibly other points of P) are moved around. One observes three things:

1. If $x \in \partial\Delta$ is not on the segment $\overline{p_2 p_4}$ then $f_3 \to \infty$ as $p_3 \to x$.

2. If $x \in \overline{p_2 p_4}$ is not equal to p_2 then $f_2 \to \infty$ as $p_3 \to x$.

3. If $x \in \overline{p_2 p_4}$ is not equal to p_4 then $f_4 \to \infty$ as $p_3 \to x$.

These three observations establish the following claim: If $f(P) \in [1, r]$, and p_1, p_2, p_4, p_5 are held fixed, then there is a compact set of positions, in the open triangle Δ where p_3 could be.

The map $P \to \{p_1, p_2, p_3, p_4, p_5\}$ gives a map from $\Sigma_n \to \Sigma_5$. There are n of these maps, depending on the choice of vertex p_1. From what we have just seen, the image of $f^{-1}[1, r]$, under each of these maps, is compact.

Suppose now that $\{P_k\}$ is a sequence of convex n-gons in $\boldsymbol{RP}^2$ such that $f(P_k) \in [1, r]$. Suppose, by induction, that we can find a sequence of projective transformations $\{T_k\}$ such that, on a subsequence, the first $m \geq 5$ points of $T_k(P_k)$ converge. Then, using the appropriate map into Σ_5 we see that, on a thinner subsequence, the $(m+1)$st points also converge. Hence, the polygons $T_k(P_k)$ converge, on a subsequence, to a fixed polygon. This proves that $f^{-1}[1, r]$ is compact.

3.3 Invariance Proof

Here we sketch the proof that $f \circ T_n = f$. This proof is different from what appears in [**S**].

Let $\Sigma_n(j)$ be the set projective classes of convex n-gons labelled by consecutive integers congruent to j mod 4. There is a canonical map from Σ_n into $\Sigma_n(1)$. A polygon whose points are labelled by integers $1, 2, 3...$ is mapped to geometrically the same polygon whose points are labelled by integers $1, 5, 9, ...$ We denote this map by $\Sigma_n \Longrightarrow \Sigma_n(1)$. We denote the inverse map by $\Sigma_n(1) \Longrightarrow \Sigma_n$.

The map $P \to P'$, formerly defined as a map on unlabelled n-gons, can naturally be interpreted either as a map $A_n : \Sigma_n(1) \to \Sigma_n(3)$ or as a map $B_n : \Sigma_n(3) \to \Sigma_n(1)$. The two interpretations are shown, for $n = 7$, in Figure 3.3. The map $T_n : \Sigma_n \to \Sigma_n$ factors in the following way:

$$\Sigma_n \Longrightarrow \Sigma_n(1) \xrightarrow{A_n} \Sigma_n(3) \xrightarrow{B_n} \Sigma_n(1) \Longrightarrow \Sigma_n.$$

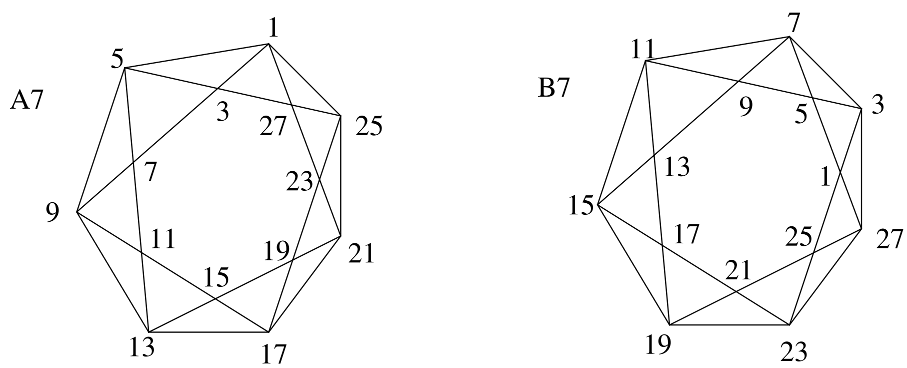

Figure 3.3.

Let E and O be the invariants defined in the previous section. To show that $f \circ T_n = f$ we will show that

$$(*) \qquad E \circ A_n = O; \quad O \circ A_n = E; \quad E \circ B_n = O; \quad O \circ B_n = E.$$

Let $P \in \Sigma_n(1)$. The vertices of P are labelled by integers congruent to 1 mod 4. We coordinatize P by the variables $(x_1, y_2, ..., x_{2n-1}, y_{2n})$, where

$$x_{\frac{i+1}{2}} = O_i(P); \quad y_{\frac{i-1}{2}} = E_i(P).$$

Here, for instance, $O_1(P)$ is the quantity, computed in the previous section, for the vertex 1. In these coordinates, $O(P) = \prod x_i$ and $E(P) = \prod y_i$.

We coordinatize $P' = A_n(P)$ by the variables

$$x'_{\frac{i-1}{2}} = O_i(P'); \quad y'_{\frac{i+1}{2}} = E_i(P').$$

We coordinatize $P'' = B_n(P')$ exactly as we coordinatized P, using variables x'' and y''.

A calculation shows that

$$x'_j = \left(\frac{1 - x_{j+2}y_{j+3}}{1 - x_{j-2}y_{j-1}}\right) y_{j+1}; \qquad y'_j = \left(\frac{1 - x_{j-3}y_{j-2}}{1 - x_{j+1}y_{j+2}}\right) x_{j-1},$$

and

$$x''_j = \left(\frac{1 - x'_{j-2}y'_{j-3}}{1 - x'_{j+2}y'_{j+1}}\right) y'_{j-1}. \qquad y''_j = \left(\frac{1 - x'_{j+3}y'_{j+2}}{1 - x'_{j-1}y'_{j-2}}\right) x'_{j+1}.$$

The identities in Equation $(*)$ follow immediately from these equations.

4 The Volume Form

4.1 Framings and Volume Forms

Suppose that $\widetilde{X}$ is a smooth manifold. A *framing* of $\widetilde{X}$ is a smoothly varying choice of basis for the tangent spaces of $\widetilde{X}$. That is, for each $x \in \widetilde{X}$ we have a basis F_x for the tangent space $T_x\widetilde{X}$. If F is a framing on $\widetilde{X}$, then F canonically determines a volume form μ_F. Namely, μ_F is the volume form which assigns the value 1, at each point, to the basis given by F.

Suppose that $\widetilde{X}_1$ and $\widetilde{X}_2$ are two smooth manifolds, equipped with framings F_1 and F_2 respectively. If $\alpha : \widetilde{X}_1 \to \widetilde{X}_2$ is a diffeomorphism, and $x_1 \in \widetilde{X}_1$ is a point, we define the matrix M_{x_1}, as follows: We have the differential map

$$d\alpha : T_{x_1}\widetilde{X}_1 \to T_{x_2}\widetilde{X}_2.$$

Here $x_2 = \alpha(x_1)$. We write out this map with respect to the bases given by F_1 and F_2. This is our matrix. We say that α is *adapted to* (F_1, F_2) if $\det(M_x) = 1$ for all $x \in \widetilde{X}_1$. Note that α is adapted to (F_1, F_2) iff the differential $d\alpha$ maps μ_{F_1} to μ_{F_2}.

Suppose now that $G : \widetilde{X} \to \widetilde{X}$ is a smooth, free group action. Here *free* means that every element of G acts with no fixed points. In this case the quotient $X = \widetilde{X}/G$ is also a smooth manifold. If $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is a smooth map which commutes with G then there is an induced map $T : X \to X$. Here is our main technical result.

Lemma 4.1 *Suppose $G : \widetilde{X} \to \widetilde{X}$ is a smooth free group action. Suppose $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is a smooth diffeomorphism which commutes with the action of G. Suppose there exists a smooth G-invariant framing F on $\widetilde{X}$ such that $\widetilde{T}$ is adapted to (F, F). Then there is a volume form μ on $X = \widetilde{X}/G$ which is preserved by the induced map $T : X \to X$.*

Proof: We begin with a fact from linear algebra. Suppose V is an n-dimensional vector space, equipped with a volume form v. Suppose $W \subset V$ is a k-dimensional subspace, equipped with a volume form w. We shall denote the quotient map $V \to V/W$ by $x \to \overline{x}$. It is an elementary fact that there is a unique volume form q on V/W such that

$$q(\overline{x}_1 \wedge \ldots \wedge \overline{x}_{n-k}) = \frac{v(x_1 \wedge \ldots \wedge x_{n-k} \wedge y_1 \wedge \ldots \wedge y_k)}{w(y_1 \wedge \ldots \wedge y_k)}.$$

Here $x_1, ..., x_{n-k} \in V$ are vectors such that $\overline{x}_1, ..., \overline{x}_{n-k}$ is a basis for V/W, and $y_1, ..., y_k$ is and basis for W.

Let $x \in X$ be a point. Let $\widetilde{x} \in \widetilde{X}$ be a point which is mapped to X under the quotient map $\widetilde{X} \to X$. Let $V = T_{\widetilde{x}}\widetilde{X}$ be the tangent space. Let L_G be the Lie algebra of left invariant vector fields on G. We fix a left invariant volume form on L_G. Each element of L_G defines a G-invariant vector field on $\widetilde{X}$. In this way, there is a canonical embedding of L_G into V. Let W be the image of L_G under this embedding, at $\widetilde{x}$. The tangent space to X at x is canonically isomorphic to the quotient V/W.

Note that V has the volume form $v = \mu_F$. Also, W has a volume form given by its identification with L_G. We use the linear algebra fact above to get a volume form μ_x on $T_xX = V/W$. This construction of μ_x does not depend on the choice of $\widetilde{x}$, because everything in sight is G-invariant. Let μ be the volume form on X which restricts to μ_x at each point $x \in X$. The naturality of our construction implies that T preserves μ. ♠

To prove Theorem 1.1 it remains to prove the Volume Lemma. Here is an outline of its proof. Let $\widetilde{\Sigma}_n$ be the space of strictly convex n-gons in $\boldsymbol{RP}^2$. Let $G = PGL_3(\boldsymbol{R})$. Note that G acts freely and smoothly on $\widetilde{\Sigma}_n$, and that $\widetilde{\Sigma}_n/G = \Sigma_n$. Let $\widetilde{T}_n : \widetilde{\Sigma}_n \to \widetilde{\Sigma}_n$ be the pentagram map, as it acts on $\widetilde{\Sigma}_n$. Note that the $\widetilde{T}_n$ induces the map $T_n : \Sigma_n \to \Sigma_n$.

We define $\widetilde{\Sigma}_n(j)$ as the space of strictly convex n-gons in $\boldsymbol{RP}^2$, whose points are labelled by consecutive integers congruent to j mod 4. Just as we factored the map T_n in §3.3, we factor $\widetilde{T}_n$ as:

$$\widetilde{\Sigma}_n \Longrightarrow \widetilde{\Sigma}_n(1) \xrightarrow{\widetilde{A}_n} \widetilde{\Sigma}_n(3) \xrightarrow{\widetilde{B}_n} \widetilde{\Sigma}_n(1) \Longrightarrow \widetilde{\Sigma}_n.$$

The double arrows indicate maps which just change the labels. The factorization here forms an obvious commuting diagram with the one given in §3.3.

Below we will construct G-invariant framings $F(1)$ and $F(3)$, respectively on $\widetilde{\Sigma}_n(1)$ and $\widetilde{\Sigma}_n(3)$. Then we will show that $\widetilde{A}_n$ is adapted to $(F(1), F(3))$. It follows from symmetry (or from a similar proof) that $\widetilde{B}_n$ is adapted to $(F(3), F(1))$. The composition $\widetilde{B}_n \circ \widetilde{A}_n$ is therefore adapted to $(F(1), F(1))$. But this composition differs from $\widetilde{T}_n$ only in the labels on the points of the polygons. So, there exists a G-invariant framing F on $\widetilde{\Sigma}_n$ such that $\widetilde{T}_n$ is adapted to (F, F). By Lemma 4.1, there exists a volume form μ_n on Σ_n which is T_n-invariant.

4.2 Unit Vector in the Hilbert Metric

Suppose that L is a line in $\boldsymbol{R}^2$, and $A, B, C \in L$ are three points, such that B separates A from C. We define

$$V(A,B,C) = \frac{(C-B)(B-A)}{C-A}.$$

Our expression requires some explanation. The quantities $C-B$, etc. are vectors, all in the same one dimensional subspace, so that it makes sense to multiply and divide them.

The Hilbert metric on the line segment $[A, C]$ is a Riemannian metric. It is just a pointwise multiple of the Euclidean metric on $[A, C]$. Thus, it makes sense to talk about the length of vectors tangent to $[A, C]$, as measured in the Hilbert metric. It is not hard to see that $V(A, B, C)$ is the tangent vector, based at B, oriented from A to C, which has unit length in the Hilbert metric on $[A, C]$. The geometric interpretation of $V(A, B, C)$ shows that it is invariant under projective transformations.

From the naturality of the construction, $V(A, B, C)$ makes sense for any three collinear points in $\boldsymbol{RP}^2$, even if the formula breaks down. The breakdown occurs if one of the points is infinite, or, more generally, if the segment $[A, C]$ intersects the line at infinity.

4.3 The Framings

To construct our framing in $\widetilde{\Sigma}_n(j)$ we need to construct, for each point in $\widetilde{\Sigma}_n(j)$, a basis for the tangent space at that point. A point in $\widetilde{\Sigma}_n(j)$ is a polygon in $\boldsymbol{RP}^2$. A tangent vector to the *point* is just a collection of n vectors in the plane, one per vertex of the *polygon*. To avoid using the word "tangent" too frequently, we will call the vectors in the plane *motion vectors*. Thus, a tangent vector to a polygon is a collection of n motion vectors. The intuition is that the collection of n motion vectors tells us how to move the polygon, to get a nearby polygon.

Suppose we are given a polygon P, a vertex p of P, and a single motion vector v. We can interpret v as a tangent vector by setting all the other $n-1$ motion vectors equal to 0. We call this process *extension*. The extension process starts with a motion vector and promotes it to a tangent vector by including it as the only nonzero vector in a collection of n motion vectors. This is what we will do in constructing our basis for the tangent space to

$\widetilde{\Sigma}_n(j)$. We will specify a collection of $2n$ motion vectors $v_1, ..., v_{2n}$. Each motion vector is extended to a tangent vector. Thus, the $2n$ motion vectors determine $2n$ tangent vectors, a basis for the tangent space.

Here is the construction for $\widetilde{\Sigma}_n(1)$. Let $P = p_1, p_5, p_9, ...$ be a polygon in $\widetilde{\Sigma}_n(1)$. Let

$$q_{j+2} = \overline{p_{j-4}p_j} \cap \overline{p_{j+4}p_{j+8}}.$$

For $j = 1, ..., n$ we define the motion vectors

$$v_{2j} = V(p_{2j+7}, p_{2j+3}, q_{2j+1}); \qquad v_{2j+1} = V(p_{2j-1}, p_{2j+3}, q_{2j+5}).$$

Two of these vectors are shown in Figure 4.1

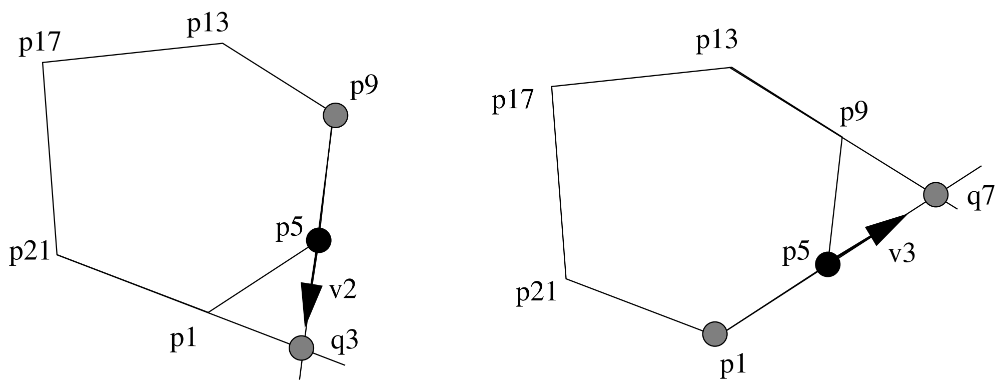

Figure 4.1

Here is the construction for $\widetilde{\Sigma}_n(3)$. Let $P = p_3, p_7, p_{11} ... \in \widetilde{\Sigma}_n(3)$. This time, we define the motion vectors

$$v_{2j-1} = V(p_{2j+5}, p_{2j+1}, q_{2j-1}); \qquad v_{2j} = V(p_{2j-3}, p_{2j+1}, q_{2j+3}).$$

Two of these vectors are shown in Figure 4.2.

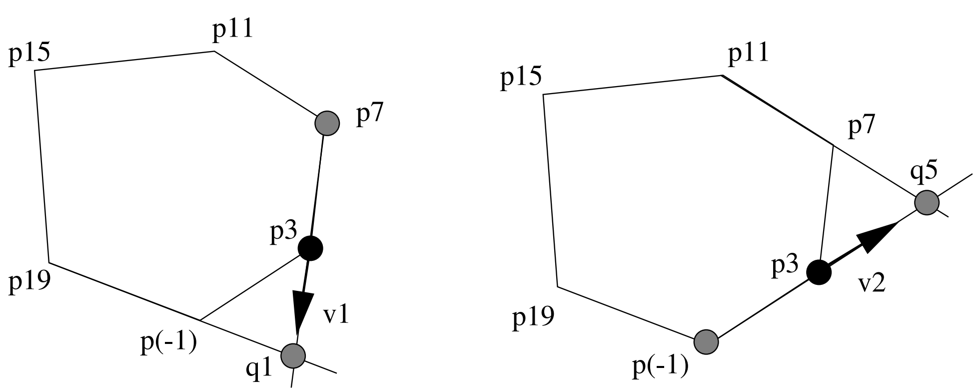

Figure 4.2

4.4 Form of The Matrix

We are interested in showing that the map $\widetilde{A}_n : \widetilde{\Sigma}_n(1) \to \widetilde{\Sigma}_n(3)$ is adapted to $(F(1), F(3))$, the pair of framings constructed in the previous section. In this section we work out the matrix $d\widetilde{A}_n$, written with respect to these two framings. For ease of exposition, we will consider the case $n = 7$.

Let λ_{ij} be the matrix entries of $d\widetilde{A}_n$. The expression λ_{ij} is a function on $\widetilde{\Sigma}_n(1)$. Figure 4.3 shows polygons P and $Q = \widetilde{A}_n(P)$. The vertices of P are labelled by $...p_9, ...$ and the vertices of Q are labelled by $...q_3, ..., q_{15}....$ Figure 4.3 shows what happens when we vary P along $-v_4$. This is to say, we keep all the vertices of P fixed, except for p_9, and we slide p_9 along the vector $-v_4$. This is indicated in the picture by an arrow emanating from p_9. The other arrows in the figure indicate the motion of the points of Q which are affected by the variation.

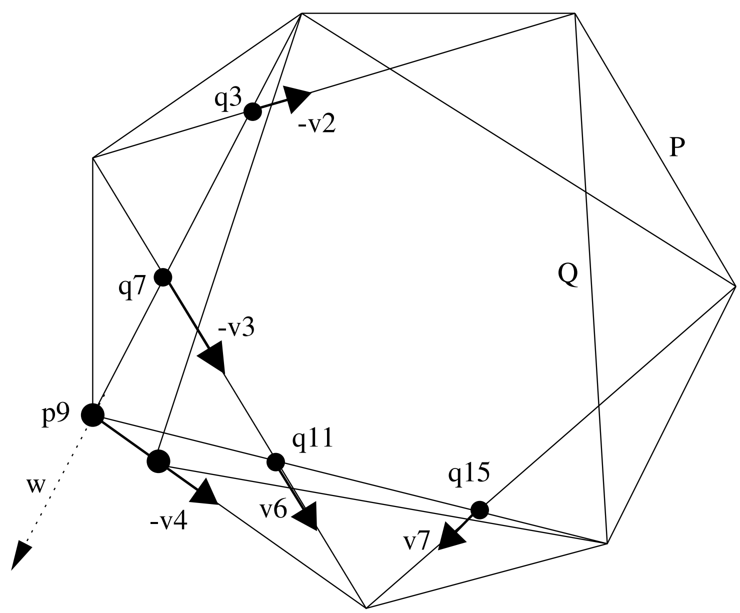

Figure 4.3

From Figure 4.3 one can see that $\lambda_{4j} > 0$ iff $j = 2, 3$, and $\lambda_{4j} < 0$ iff $j = 6, 7$. Likewise $\lambda_{5j} < 0$ iff $j = 2, 3$ and $\lambda_{5j} > 0$ iff $j = 6.7$. Define

$$\Lambda^{9,5} = \begin{bmatrix} \lambda_{42} & \lambda_{52} \\ \lambda_{43} & \lambda_{53} \end{bmatrix} \qquad \Lambda^{9,13} = \begin{bmatrix} \lambda_{46} & \lambda_{56} \\ \lambda_{47} & \lambda_{57} \end{bmatrix}$$

Variation along the vector w, which is a linear combination of v_4 and v_5, does not move q_3 or q_7. This is to say that the linear transformation represented by $\Lambda_{9,5}$ has a kernel. Hence $\det(\Lambda^{9,5}) = 0$. Likewise, $\det(\Lambda^{9,13}) = 0$. The same picture occurs at each vertex. Shifting the indices in a more or less obvious way:

$$d\widetilde{A}_7 = \begin{bmatrix} 0 & \Lambda^{5,1} & 0 & 0 & 0 & 0 & \Lambda^{25,1} \\ \Lambda^{1,5} & 0 & \Lambda^{9,5} & 0 & 0 & 0 & 0 \\ 0 & \Lambda^{5,9} & 0 & \Lambda^{13,9} & 0 & 0 & 0 \\ 0 & 0 & \Lambda^{9,13} & 0 & \Lambda^{17,13} & 0 & 0 \\ 0 & 0 & 0 & \Lambda^{13,17} & 0 & \Lambda^{21,17} & 0 \\ 0 & 0 & 0 & 0 & \Lambda^{17,21} & 0 & \Lambda^{25,21} \\ \Lambda^{1,25} & 0 & 0 & 0 & 0 & \Lambda^{21,25} & 0 \end{bmatrix}. \qquad (1)$$

Each 0 stands for the 2×2 zero matrix. The individual 2×2 blocks $\Lambda^{i,j}$ have determinant 0. The general pattern should be clear from the case $n = 7$.

4.5 Computing a Block in the Matrix

Let $x_1, y_2, \ldots, x_{2n-1}, y_{2n}$ be the invariant coordinates for P, defined in §3.3. We also define

$$z_{ij} = \frac{1}{1 - y_i x_j}.$$

The entire structure of $d\widetilde{A}_n$ can be deduced from symmetry and from the formula

$$(*) \qquad \Lambda^{9,13} = \begin{bmatrix} -x_5 z_{45} & z_{45} \\ -x_5 y_6 z_{89} & y_6 z_{89} \end{bmatrix}.$$

The fact that the $\det \Lambda^{9,13} = 0$ implies that the formulae for any three entries determines the fourth. We will compute $\lambda_{46} = -x_5 z_{45}$ and omit the other two calculations, which are similar. We point out that one can verify these calculations, numerically, on the computer.

Referring to Figure 4.5, let $|XY|$ be the Euclidean length of the segment having endpoints X and Y. In particular, let $\epsilon = |BC|$ and $\delta = |DE|$

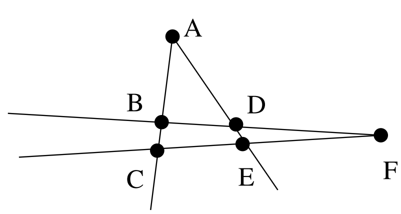

Figure 4.5

Basic plane geometry gives

$$\delta = \frac{|AD||FD|}{|AB||FB|}\, \epsilon + O(\epsilon^2).$$

Figure 4.4 below shows the same polygons as does Figure 4.3. The vertex p_9' represents a small perturbation of p_9, along the motion vector $-v_4$. The vertex q_{11}' represents a small perturbation of q_{11} along the motion vector v_6.

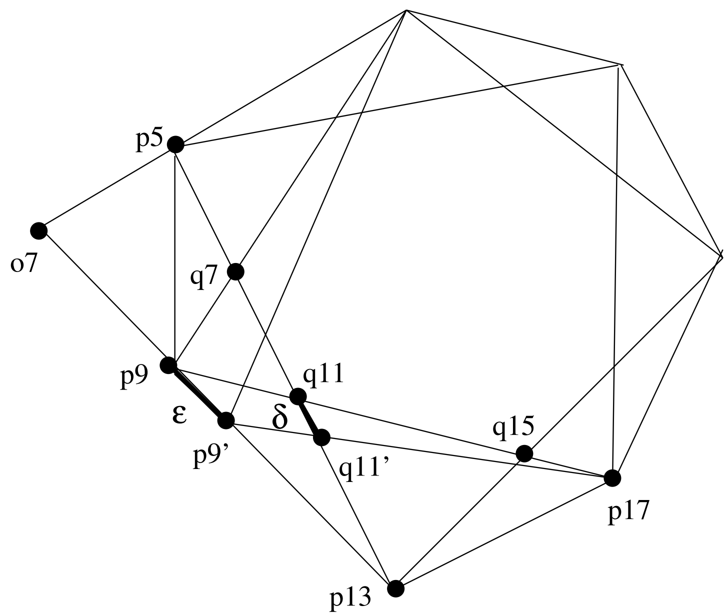

Figure 4.4

Define

$$\alpha = \frac{-|p_{13} - o_7|}{|o_7 - p_9||p_9 - p_{13}|};\ \ \beta = \frac{|p_{13} - q_{11}||p_{17} - q_{11}|}{|p_{13} - p_9||p_{17} - p_9|};\ \ \gamma = \frac{|p_{13} - q_7|}{|p_{13} - q_{11}||q_{11} - q_7|}.$$

We set $\epsilon = |p_9 - p_9'|$ and $\delta = |q_{11} - q_{11}'|$. Using the definitions of our framing for the polygon P, we see that $p_9' - p_9$ is $-A\epsilon + O(\epsilon^2)$ times v_4. From the geometric fact above, $\delta = B\epsilon + O(\epsilon^2)$. Using the definition of our framing for P', we see that $q_{11}' - q_{11}$ is $C\delta + O(\delta^2)$ times v_6. Letting $\epsilon \to 0$, and recalling that $\lambda_{46} < 0$, we have

$$\lambda_{46} = \beta\gamma/\alpha = -\frac{|p_9 - o_7||p_{13} - q_7||q_{11} - p_{17}|}{|o_7 - p_{13}||q_7 - q_{11}||p_{17} - p_9|}.$$

A short calculation identifies the expression above with $-x_5 z_{45}$.

4.6 Computing the Determinant

Here is the arrangement of the blocks $\Lambda^{5,9}$, $\Lambda^{13,9}$, $\Lambda^{9,5}$ and $\Lambda^{9,13}$:

$$\begin{array}{cccccc} & & \lambda_{42} & \lambda_{52} & & \\ & & \lambda_{43} & \lambda_{53} & & \\ \lambda_{24} & \lambda_{34} & 0 & 0 & \lambda_{64} & \lambda_{74} \\ \lambda_{25} & \lambda_{35} & 0 & 0 & \lambda_{65} & \lambda_{75} \\ & & \lambda_{46} & \lambda_{56} & & \\ & & \lambda_{47} & \lambda_{57} & & \end{array}.$$

We define

$$V_2 = \frac{\det \begin{bmatrix} \lambda_{42} & \lambda_{52} \\ \lambda_{47} & \lambda_{57} \end{bmatrix}}{\lambda_{42}\lambda_{57}}; \qquad H_2 = \frac{\det \begin{bmatrix} \lambda_{24} & \lambda_{74} \\ \lambda_{25} & \lambda_{75} \end{bmatrix}}{\lambda_{24}\lambda_{75}}; \qquad D_2 = \lambda_{56}\lambda_{65}.$$

We define H_{2+i}, V_{2+i} and D_{2+i} by adding $2i$ to all of the indices in the λ_{jk}. For instance, $D_1 = \lambda_{34}\lambda_{43}$. Equation $(*)$ gives

$$H_2 = V_2 = 1/z_{45}; \qquad D_2 = z_{45}z_{67}.$$

Hence $\prod H_i V_i D_i = 1$. To finish the proof of the Volume Lemma we show

Lemma 4.2 $\det(d\widetilde{A}_n) = \prod_{i=1}^{n} H_i V_i D_i$.

Proof: For ease of exposition, we take $n = 6$. The polynomial $Z = \det(d\widetilde{A}_6)$ consists of $n!$ signed monomials. The monomials in Z are only nonzero when all variables have been chosen from the nonzero 2×2 blocks. Say that a monomial in Z is *bad* if it contains two variables picked from the same nonzero block, and otherwise *good*. The bad monomials cancel in pairs, since the determinant of each block is 0.

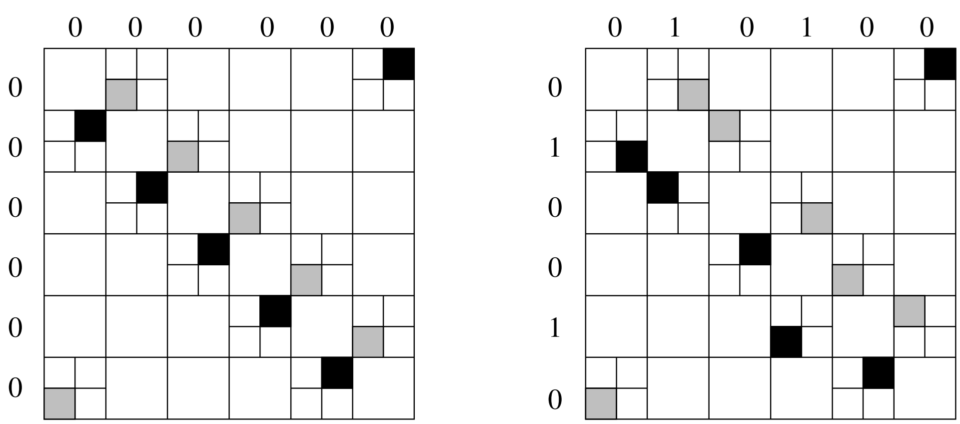

Figure 4.6

Figure 4.6 illustrates a coding for the good monomials. Each good monomial is specified by choosing one variable arbitrarily from each of the 6 above-diagonal blocks $\Lambda^{j,j-4}$. These choices, which are represented by lightly colored squares, uniquely determine the variables in the below-diagonal blocks $\Lambda^{j,4+j}$, which are represented by black squares. The monomials are signed, so that Z is a positive sum over all these monomials.

We can encode each one of these monomials by a pair of binary strings (a,b). Both a and b have length 6. The 1 bits in a indicate the columns in which the light shaded square is on the right half of the 2×2 block. The 1 bits in b indicate the rows in which the light shaded square are on the top half of the 2×2 block. For instance, the first picture is encoded by $(000000, 000000)$. The second picture is encoded by $(010100, 010010)$. Note that $(000000, 000000) = \prod D_k$.

If a has a 0 in the kth position, let a_k be the string obtained by changing this bit to a 1. For instance $(010010)_3 = (011010)$. We do not define a_k if a has a 1 in the kth position. We make the same definition for b. We have the following basic identity, which uses the fact that $\det \Lambda^{ij} = 0$.

$$(*) \qquad (a,b) + (a_k,b) = (a,b)H_k; \qquad (a,b) + (a,b_k) = (a,b)V_k.$$

Let $*$ stand for either a 0 or a 1. Let S_{ij} be the set of monomials of the form $(*...0, *...0)$, such that there are i copies of $*$ in the first slot, and j copies of $*$ in the second slot. For example, S_{25} consists of the set of all monomials having the form $(**0000, *****0)$. Obviously, S_{66} is the set of all monomials.

Formula $(*)$ gives us $\det(d\widetilde{A}_6) = \sum_{S_{66}} (a,b) =$

$$H_5 \sum_{S_{56}} (a,b) = H_5 H_4 \sum_{S_{46}} (a,b) = ... = \prod H_i \sum_{S_{06}} (a,b) =$$

$$V_5 \prod H_i \sum_{S_{05}} (a,b) = ... = \prod V_i \prod H_i \sum_{S_{00}} (a,b) = \prod H_i V_i D_i. \ \spadesuit$$

5 References

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[**H**] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, 1950.

[**S**] R Schwartz, *The Pentagram Map*, Journal of Experimental Mathematics, Vol 1., 1992.