The Density of Shapes in Three Dimensional Barycentric Subdivision

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1 Introduction

The barycentric subdivision of an $n$-dimensional simplex $\Delta$ is a certain collection of $(n + 1)!$ smaller $n$-simplices whose union is $\Delta$. The construction is defined by induction on $n$. If $n = 0$ then $\Delta$ is a single point, and the barycentric subdivision of $\Delta$ is this same point. In general, if $\Delta'$ is one of the simplices in the barycentric subdivision of $\Delta$ then $\Delta'$ is the convex hull of a set of the form $v \cup F'$, where $v$ is the center of mass of $\Delta$—i.e. the barycenter—and $F'$ is one of the simplices in the barycentric subdivision of one of the top dimensional faces $F$ of $\Delta$. See [S, p. 123] or §2 below for more details.

Consider the following dynamical process: Start with an $n$-simplex $\Delta$ and barycentrically subdivide $\Delta$ into simplices $\Delta_1, \ldots, \Delta_{(n+1)!}$. Next, subdivide $\Delta_j$ into simplices $\Delta_{j1}, \ldots, \Delta_{j(n+1)!}$, for each $j$. And so forth. This process produces an infinite collection $C$ of simplices. A natural question is: Does $C$ consist of a dense set of shapes? By shape we mean a simplex modulo similarities.

In [BBC] this question was raised and answered in the 2-dimensional case. Part of their idea works in all dimensions. Let $T$ be the collection of matrices of the form $T = L/|\det(L)|^{1/m}$, where $L$ is the linear part of an affine map from $\Delta$ to a member of $C$. The affine naturality of barycentric subdivision

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forces $\mathcal{T}$ to be a semigroup of $SL_n(\mathbb{R})$, the group of $n \times n$ determinant-1 matrices.

When $n = 2$, a calculation in [BBC] shows that $\mathcal{T}$ contains some infinite order elliptic elements. (In general, an elliptic element of $SL_n(\mathbb{R})$ is a matrix which generates a subgroup having compact closure, which happens iff the matrix is diagonalizable over $\mathbb{C}$ with all eigenvalues unit complex numbers.) The set of powers of an infinite order elliptic element is dense in a compact subgroup of $SL_2(\mathbb{R})$ and these dense sets are used to show that $\mathcal{T}$ is dense in $SL_2(\mathbb{R})$. Hence, in the 2-dimensional case, $\mathcal{T}$ contains a dense set of triangles.

Using a computer search, which we detail in the next section, we found some infinite order elliptic elements in the 3-dimensional case. This seems like a lucky accident, because the set of elliptic elements in $SL_n(\mathbb{R})$ has measure zero for $n \geq 3$. Using these elliptic elements, some basic Lie group theory, and Mathematica [W], we prove

**Theorem 1.1** The 3-dimensional barycentric subdivision process produces a dense set of shapes of tetrahedra.

A similar computer search failed to turn up any elliptic elements in the case $n = 4$, though we certainly would have liked to make a deeper search using a more powerful computer. We think that the density result should be true in all dimensions, whether or not $\mathcal{T}$ contains elliptic elements.

I would like to thank Bill Goldman for some interesting discussions about Lie groups and Lie algebras.

## 2 The Proof

Here we give a concrete description of barycentric subdivision in the 3-dimensional case. Let $\Delta$ be the convex hull of points $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$. Let $S_4$ be the group of permutations of the set $\{0, 1, 2, 3\}$. Given $\sigma = (i_0, i_1, i_2, i_3) \in S_4$, let $c_k$ be the center of mass of the points $v_{i_0}, \ldots, v_{i_k}$. Let $\Delta_\sigma$ be the convex hull of the points $c_0, c_1, c_2, c_3$. The union $\bigcup_{\sigma \in S_4} \Delta_\sigma$ is the barycentric subdivision of $\Delta$.

To begin our dynamical process, we take the initial tetrahedron $\Delta$ to be the convex hull of the vertices $e_0, e_1, e_2, e_3$. Here $e_0$ is the origin and $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Let $A_\sigma$ be the affine map such that $A_\sigma(e_k) = c_k$ for $k = 0, 1, 2, 3$. Let $L_\sigma$ be the linear part of $A_\sigma$. Finally, let
\[ T_\sigma = L_\sigma / |\det(L_\sigma)|^{1/3}. \] By construction, \( A_\sigma(\Delta) = \Delta_\sigma \) and therefore \( T_\sigma \subseteq T \), the semigroup discussed in §1.

We order the 24 elements of \( S_4 \) lexicographically. For instance \( \sigma_1 = (0123) \) and \( \sigma_2 = (0132) \). We define

\[ F(i,j,k) = T_{\sigma_k} \circ T_{\sigma_j} \circ T_{\sigma_i}. \]

Say that the triple \((i,j,k)\) is good if \( F(i,j,k) \) is an infinite order elliptic element. A computer search reveals 39 good sequences. Here is the list, modulo cyclic permutations:

\[(2,15,19); (5,8,23); (5,19,18); (5,20,16); (7,17,8); (8,18,9); (8,18,20);
(8,23,16); (9,19,23); (15,19,16); (16,16,19); (16,19,18); (19,23,20)\]

We had hoped to see a divine pattern in this list, but did not.

Our density proof uses only the elements

\[ S = F(23,20,19); \quad M_1 = F(5,20,16); \quad M_2 = F(20,16,5). \]

Another triple of elements from the list would probably work just as well. In the appendix we include a short Mathematica program which computes:

\[
S = \frac{1}{24} \begin{bmatrix} 54 & 48 & 39 \\ -6 & -32 & -35 \\ -78 & -32 & -23 \end{bmatrix};
\]

\[
M_1 = \frac{1}{72} \begin{bmatrix} -60 & -68 & -27 \\ 36 & 12 & 81 \\ -60 & 4 & 27 \end{bmatrix}; \quad M_2 = \frac{1}{24} \begin{bmatrix} 18 & 12 & 21 \\ -54 & -68 & -71 \\ 54 & 52 & 43 \end{bmatrix}.
\]

**Lemma 2.1** \( S, M_1 \) and \( M_2 \) are infinite order elliptic elements of \( SL_3(\mathbb{R}) \).

**Proof:** The eigenvalues of \( S \) and \( M_j \) respectively are \( \{1, \alpha, \overline{\alpha}\} \) and \( \{1, \beta, \overline{\beta}\} \), where \( \alpha = -25/48 + i\sqrt{1676}/48 \) and \( \beta = -31/48 + i\sqrt{1343}/48 \). Both \( \alpha \) and \( \beta \) have norm 1, so \( S \) and \( M_j \) are elliptic. If \( S \) had finite order then \( \alpha \) would be a primitive \( n \)th root of unity for some \( n \). Then \( \alpha \) would have \( \phi(n) \) distinct Galois conjugates, where \( \phi \) is the Euler phi-function. Since \( \alpha \) is a quadratic irrational, we have \( \phi(n) = 2 \). The forces \( n \leq 6 \). Clearly, \( \alpha \) is not an \( n \)th root of unity for \( n \leq 6 \). Hence \( S \) has infinite order. The same argument works
Let \( \langle S \rangle \) be the closure of the semigroup generated by \( S \). Since \( S \) is infinite order elliptic, \( \langle S \rangle \) is a closed 1-parameter compact subgroup. Let \( G \subset SL_3(\mathbb{R}) \) be the closed subgroup generated by the 8 compact subgroups \( G_{ij} = M_i^j \langle S \rangle M_i^{-j} \). Here \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3, 4\} \).

**Lemma 2.2** \( G = SL_3(\mathbb{R}) \).

**Proof:** The lie algebra to \( SL_3(\mathbb{R}) \) is \( \mathfrak{sl}_3(\mathbb{R}) \), the space of traceless \( 3 \times 3 \) matrices. Below we will justify the claim that

\[
\mathfrak{s} = \begin{bmatrix} 70 & 54 & 57 \\ -114 & -107 & -104 \\ 18 & 52 & 37 \end{bmatrix} \in \mathfrak{sl}_3(\mathbb{R})
\]

generates \( \langle S \rangle \). By this we mean that

\[ \langle S \rangle = \{ \exp(t\mathfrak{s}) | t \in \mathbb{R} \}. \]

For \( i \) and \( j \) as above we define \( g_{ij} = M_i^j \mathfrak{s} M_i^{-j} \). By construction

\[ G_{ij} = \{ \exp(tg_{ij}) | t \in \mathbb{R} \}. \]

Let \( \mathfrak{G} \) be the vector space spanned by the 8 vectors \( g_{ij} \).

For any lie algebra vectors \( \mathfrak{a} \) and \( \mathfrak{b} \) we have the well known formula

\[
\exp(\mathfrak{a} + \mathfrak{b}) = \lim_{k \to \infty} (\exp(\mathfrak{a}/k) \cdot \exp(\mathfrak{b}/k))^k
\]

(See [FH, Exercise 8.38].) This formula easily implies that \( \exp(\mathfrak{G}) \subset G \). Since \( \dim(\mathfrak{sl}_3(\mathbb{R})) = 8 \), all we need to prove is that \( \dim(\mathfrak{G}) = 8 \). There is a natural map \( P : \mathfrak{sl}_3(\mathbb{R}) \to \mathbb{R}^8 \). We simply string out the coordinates of a trace-zero matrix \( \mathfrak{g} \), leaving off \( \mathfrak{g}(3,3) \). It is easy to see that \( P \) is a vector space isomorphism. Let \( M \) be the \( 8 \times 8 \) matrix whose rows are \( P(g_{ij}) \). We compute

\[
\det(M) = \frac{1574679337686718881331462994390117}{159739999685311463424} \neq 0.
\]

This is only possible if the vectors \( P(g_{ij}) \) span \( \mathbb{R}^8 \). ♠
Let $\overline{T}$ be the closure of $T$ in $SL_3(\mathbb{R})$. By construction $\langle S \rangle \subset T$. Since $M_j$ is infinite order elliptic element, $M_j^{k_j} \in \overline{T}$ for all relevant $i$ and $j$. Therefore the group $G_{ij}$ is contained in the semigroup $\overline{T}$. This implies that $G \subset \overline{T}$. But $G = SL_3(\mathbb{R})$. Therefore $T$ is dense in $SL_3(\mathbb{R})$. Our theorem follows immediately from this.

Our only piece of unfinished business is to justify the formula for $s$. By computing the eigenspaces of $S$ we find that the matrix

$$U = \begin{bmatrix}
-21 & 0 & 2 \\
-34 & -1 & -3 \\
58 & 2 & 0
\end{bmatrix}$$

conjugates $S$ to block triangular form:

$$U^{-1}SU = \begin{bmatrix}
1 & 0 \\
0 & B
\end{bmatrix}; \quad B = \frac{1}{48} \begin{bmatrix}
-14 & -60 \\
30 & -36
\end{bmatrix}.$$ 

Note that $B \in SL_2(\mathbb{R})$ is infinite order elliptic. Let $\langle B \rangle$ be the closure of the group generated by $B$. We claim that the matrix

$$b = 48B - 24 \text{ trace}(B)I = \begin{bmatrix}
11 & -60 \\
30 & -11
\end{bmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

generates $\langle B \rangle$ in the sense that $\langle B \rangle = \{\exp(tb) | t \in \mathbb{R}\}$. To prove this, we note that $b$ and $B$ commute, when multiplied together as matrices. Hence, for any $t \in \mathbb{R}$ the element $\beta_t = \exp(tb)$ commutes with any element of $\langle B \rangle$. As is well known $SL_2(\mathbb{R})$ acts isometrically on the hyperbolic plane $H^2$ by linear fractional transformations. The group $\langle B \rangle$, which consists entirely of elliptic elements, acts as the group of isometric rotations about some fixed point $x \in H^2$. Since $\beta_t$ commutes with all elements of $\langle B \rangle$, it must also act as an isometric rotation about $x$. Hence $\beta_t \subset \langle B \rangle$ for all $t$. Our claim now follows easily.

Since $b$ generates $\langle B \rangle$,

$$s = U \begin{bmatrix}
0 & 0 \\
0 & b
\end{bmatrix} U^{-1}$$

generates $\langle S \rangle$ in the sense of Lemma 2.1. Expanding out this product gives the formula for $s$. 

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3 Appendix: A Mathematica File

We refer the reader to [W] for details on the implementation of Mathematica. A copy of this file produced our calculations.

\[
e[0] = \{0, 0, 0\}; \quad e[1] = \{1, 0, 0\}; \quad e[2] = \{0, 1, 0\}; \quad e[3] = \{0, 0, 1\};
\]
\[
S4 = \text{Permutations}\{\{0, 1, 2, 3\}\};
\]
\[
T[n_] := \langle \text{sigma} = S4[[n]]\rangle;
\]
\[
c0 = (e[\text{sigma}[\{1\}]])/1;
\]
\[
c1 = (e[\text{sigma}[\{1\}]] + e[\text{sigma}[\{2\}]])/2;
\]
\[
c2 = (e[\text{sigma}[\{1\}]] + e[\text{sigma}[\{2\}]] + e[\text{sigma}[\{3\}]])/3;
\]
\[
c3 = (e[\text{sigma}[\{1\}]] + e[\text{sigma}[\{2\}]] + e[\text{sigma}[\{3\}]] + e[\text{sigma}[\{4\}]])/4;
\]
\[
L = \text{Transpose}\{c1 - c0, c2 - c0, c3 - c0\};
\]
\[
L/\text{Power}[\text{Abs}[\text{Det}[L]], 1/3]\]
\[
F[a_, b_, c_] := \text{Simplify}[T[a].T[b].T[c]]
\]
\[
S = F[23, 20, 19]; \quad M1 = F[5, 20, 16]; \quad M2 = F[20, 16, 5];
\]
\[
s = \{\{70, 54, 57\}, \{-114, -107, -104\}, \{18, 52, 37\}\}
\]
\[
U = \{\{-21, 0, 2\}, \{-34, -1, -3\}, \{58, 2, 0\}\}
\]
\[
\text{Ad}[x_\_, y_ \_] := x.y.\text{Inverse}[x];
\]
\[
g11 = \text{Ad}[M1, s];
\]
\[
g12 = \text{Ad}[M1.M1, s];
\]
\[
g13 = \text{Ad}[M1.M1.M1, s];
\]
\[
g14 = \text{Ad}[M1.M1.M1.M1, s];
\]
\[
g21 = \text{Ad}[M2, s];
\]
\[
g22 = \text{Ad}[M2.M2, s];
\]
\[
g23 = \text{Ad}[M2.M2.M2, s];
\]
\[
g24 = \text{Ad}[M2.M2.M2.M2, s];
\]
\[
P[x_ \_] := \text{Take}[\text{Flatten}[x], 8]
\]
\[
M = \{P[g11], P[g12], P[g13], P[g14], P[g21], P[g22], P[g23], P[g24]\}
\]
\[
\text{Det}[M]
\]
4 References


