Rotation Codings and X-Ray Dynamics

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This note describes a nice thing I discovered experimentally about the binary sequences associated to rotations of the circle. I'm asking around to see if this is a new result.

1. Rotation Codings: Let $R \in (0,1/2)$ be irrational and let $x \in (0,1/2]$. Define the binary sequence $\{a_n\}$ as follows.

$$a_n = 1 \quad \Longleftrightarrow \quad Rn \bmod 1 \in [0,x). \tag{1}$$

When $x = R$, the resulting sequence is called a *Sturmian Sequence*. When $x = 1/2$, the sequence is called a *Rote sequence*. The general case is (I think) called a *rotation coding*. These sequences are well-studied, especially the Sturmian ones.

2. Derived Ternary Sequence: Given a binary sequence, one can record the sizes of the gaps of 0's between consecutive 1's. For instance,

$$101000001000110000100 1... \quad \rightsquigarrow \quad 153042...$$

As is well known, for rotation codings, at most 3 different numbers arise in this "gap sequence". To tidy things up, we replace the smallest number with 1, the next smallest with 2, and the largest with 3. We call this the *ternary sequence derived* from the rotation coding. For example, the rotation coding

1000010000000100000001000010000000100000000000010000000100001...

has the gap sequence $4,7,7,4,7,12,7,4....$ Changing $(4,7,12) \to (1,2,3)$ gives us 12212321... as the derived ternary sequence.

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3. The Triangle Path: Let $\Delta \subset \boldsymbol{R}^3$ denote the equilateral triangle consisting of those points (x_1, x_2, x_3) such that

$$x_1 + x_2 + x_3 = 1; \qquad x_1, x_2, x_3 \geq 0. \tag{2}$$

Fix R, as above. Let $S_R(x)$ denote the rotation coding based on (R, x). Let $S'_R(x)$ denote the derived ternary sequence. Define

$$\delta_R(x) = (x_1, x_2, x_3); \qquad x_j = \text{density of}(j) \quad \text{in} \ S'_R(x). \tag{3}$$

Pat Hooper and Anatole Katok both pointed out to me that the dynamical system defined by (R, x) is equivalent to a 3-interval interval exchange transformation. The ternary sequence describes the visits to the three intervals of the IET for a suitably chosen orbit point. By ergodicity, the function $\delta_R(x)$ is just the triple of proportions of lenghs of intervals. This gives a way to get an explicit formula for $\delta_R(x)$.

Consider the image of the path $x \to \delta_R(x)$, namely

$$\Gamma_R = \bigcup_{x \in (0,1/2]} \delta_R(x) \tag{4}$$

We're interested in understanding the geometry of this path.

4. Basic Properties: It turns out that Γ_R is a discontinuous union of line segments. The discontinuities occur only at $\partial\Delta$, and they correspond to the 2-gap rotation codings. We think of the parameterization as going from $x = 1/2$ to $x = 0$. As $x \to 0$, the speed of the parameterization tends to ∞, and one traces out an infinite number of line segments. In the quadratic-irrational case, these line segments repeat in a periodic fashion, though they are traced out at ever increasing speeds. That is, the path is periodic, modulo changing the speed of the parameterization. We will show pictures of these paths in a moment.

5. Symmetrization: Before we show pictures, we will make the picture nicer by symmetrizing. Let S_3 denote the order 6 dihedral symmetry group that acts on Δ. We define

$$\widehat{\Gamma}_R = \bigcup_{\sigma \in S_3} \sigma(\Gamma_R) \tag{5}$$

6. Examples: In our pictures below, $\widehat{\Gamma}_R$ is the union of the white and blue line segments and Γ_R is the union of the white line segments. The yellow triangle is Δ. We have also shown a red triangle, in which Δ is inscribed in the obvious way.

Figure 1 shows the simplest possible case,

$$R = \frac{3 - \sqrt{5}}{2}$$

In this case, Γ_R is a single white line segment and $\widehat{\Gamma}_R$ is the union of two inscribed triangles. Notice that each line segment, when extended to a line, contains a vertex of the red triangle.

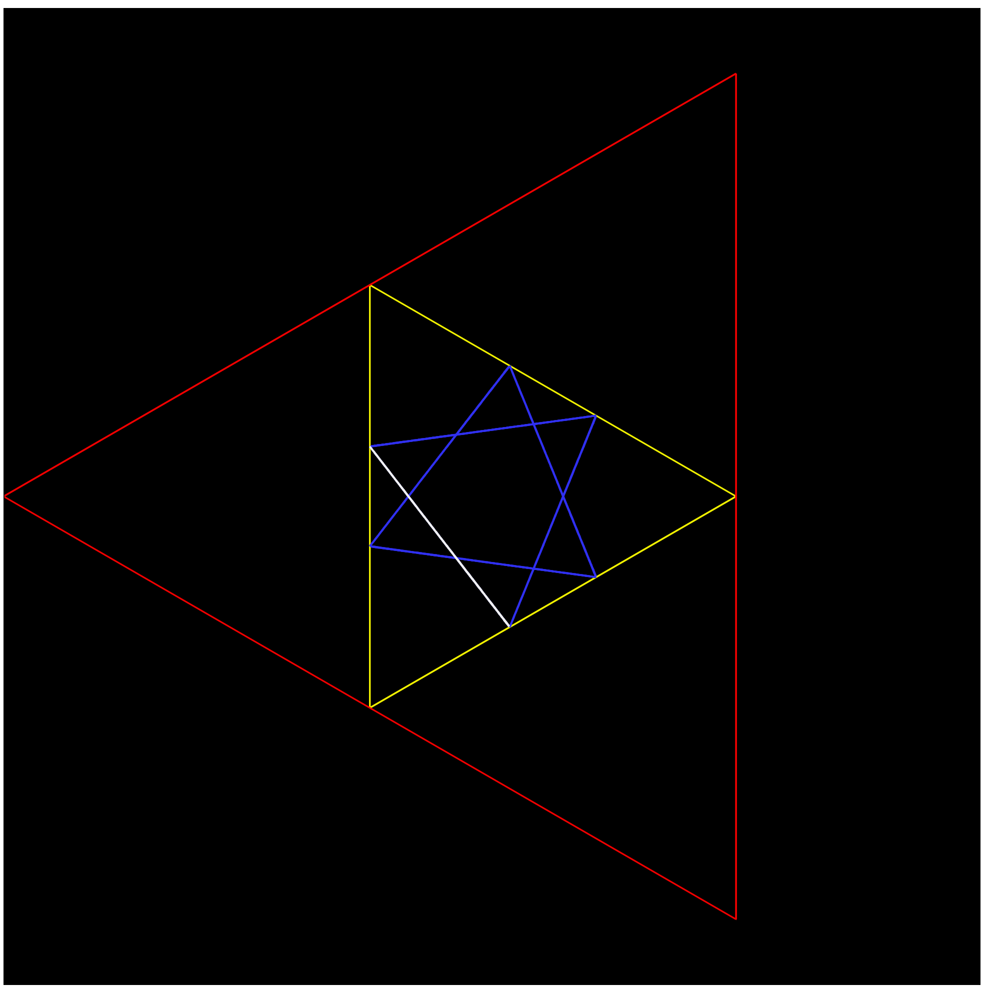

Figure 1: The picture for $R = (3 - \sqrt{5})/2$.

Figure 2 shows the next simplest example,

$$R = \frac{2-\sqrt{2}}{2}$$

Notice again, that each line segment, when extended to a line, contains a vertex of the red triangle.

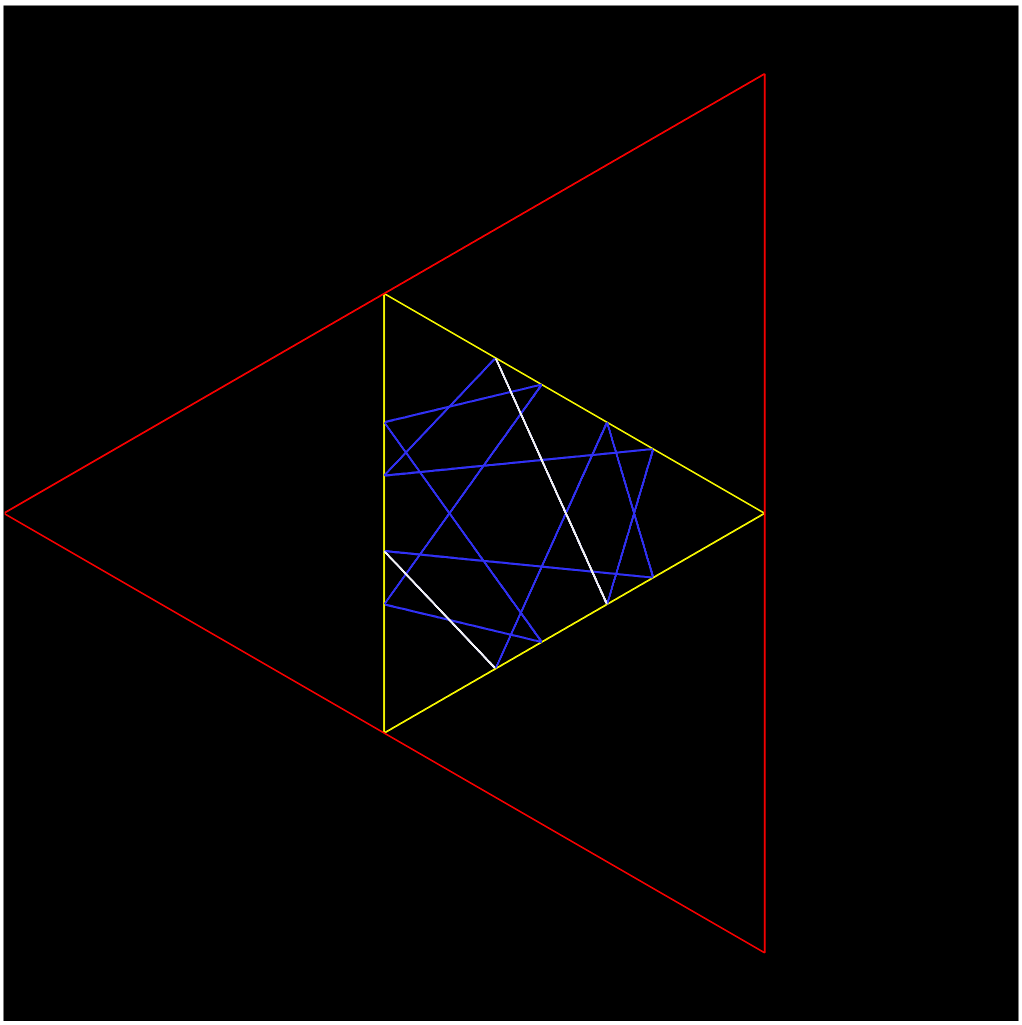

Figure 2: The picture for $R = (2-\sqrt{2})/2$.

Figure 3 shows a more complicated example,

$$R = \frac{2-\sqrt{3}}{2}$$

This time, we have zoomed in on the picture so that the red triangle is not entirely visible.

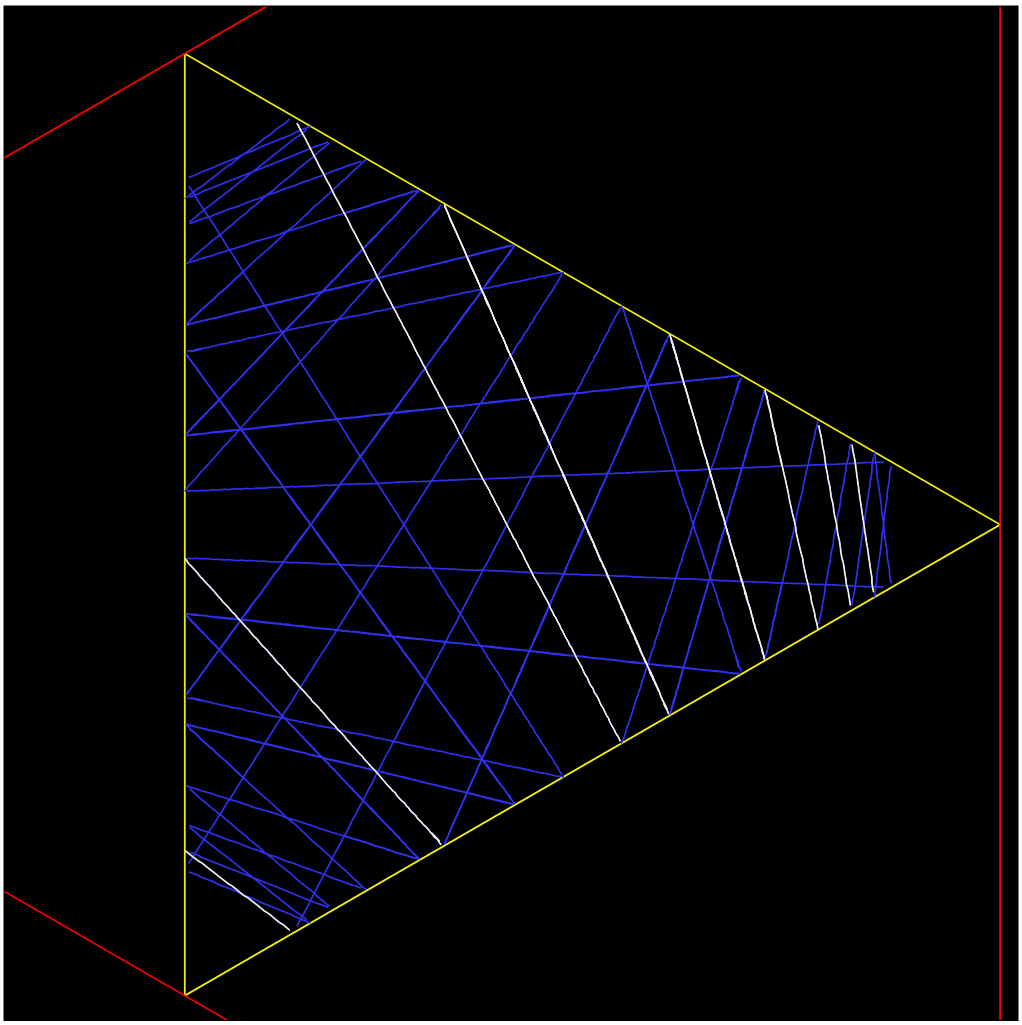

Figure 3: The picture for $R = (2-\sqrt{3})/2$.

$\widehat{\Gamma}_R$ looks somewhat like a billiard path in the yellow triangle, but this is not quite the case. It is a path based on a different kind of dynamical system.

7. X-Ray dynamics: The triangle Δ is inscribed in an obvious triangle Δ' that is twice as large. This is the red triangle in the figures above. Figure 4 shows a dynamical system one can define in the interior of Δ. Technically, the dynamical system is defined on $\partial\Delta$ (minus the vertices.) Let $p_0 \in \partial\Delta$ be some point, contained in the edge s_0. Let q_0 be the vertex of Δ' closest to p_0. Now we define

$$p_1 = \overrightarrow{q_0 p_0} \cap (\partial\Delta - s_0), \tag{6}$$

The dynamical system is then $p_0 \to p_1 \to p_2 ...$

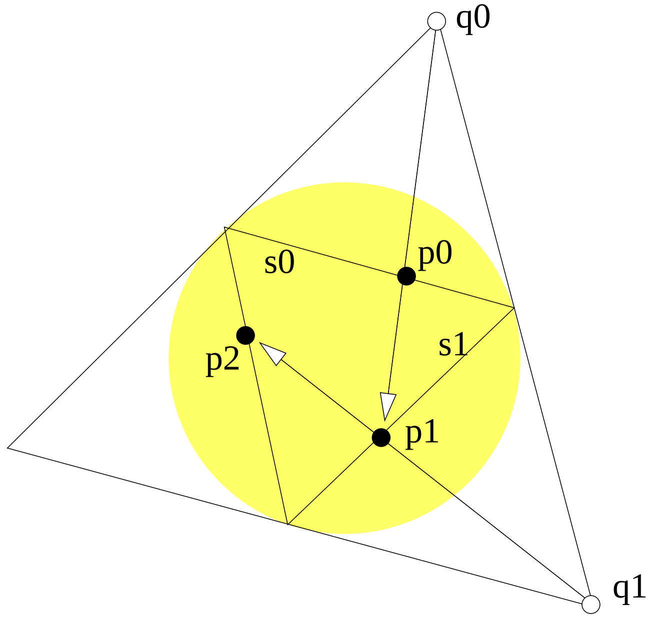

Figure 3: The X-ray dynamics in Δ.

One gets a polygon by connecting consecutive points in the orbit. We call such a polygon an *X-Ray polygon.*

The yellow disk in Figure 3 shows the connection to hyperbolic geometry. If one works in the Klein model, then Δ becomes an ideal triangle. One can consider a kind of billiards in the ideal triangle where one always bounces off the sides at right angles. These paths are the same as the X-ray polygons.

8. The Main Observation: $\widehat{\Gamma}_R$ *is contained in the* S_3*-orbit of a single X-ray polygon.*

9. Locally projective maps: To get a more complete picture, we would like to say something about the parametrization of $\widehat{\Gamma}$. As a prelude to this discussion, we need to make a formal definition.

Let L be an open line segment. Let $\phi : L \to \boldsymbol{R}$ be a map. We call ϕ *locally projective* if ϕ is the restriction of a real projective transformation. This is equivalent to the statement that ϕ preserves the cross ratio of all 4-tuples of points on which ϕ is defined.

Suppose now that $\phi : L \to \boldsymbol{R}^2$ is a map and $\pi : \boldsymbol{R}^2 \to \boldsymbol{R}$ is a linear functional. We call ϕ *locally projective relative to* π if the composition $\pi \circ \phi : L \to \boldsymbol{R}$ is locally projective.

10. Global Projectivity We say first of all that the curve $x \to \delta(x)$ is locally projective (onto its image) away from the points that map to $\partial\Delta$. However, we can make the stronger statement that our curve is actually everywhere locally projective, provided we interpret things the right way.

Recall that $\delta : [1/2, 0) \to \Gamma$ is the parameterization of Γ. Rather than consider the curve $x \to \delta(x)$, we can consider the curve

$$x \to \widehat{\delta}(x) = \sigma_x \circ \delta(x), \tag{7}$$

where σ_x is a suitable permutation of the coordinates. Assuming that $\sigma_{1/2}$ is the identity, there is a unique choice of σ_x such that $\widehat{\delta}(x)$ is smooth sway from $\partial\Delta$. In this case, $\widehat{\delta}$ necessarily traces out a single X-ray polygon.

We are interested in 2 consecutive segments of $\widehat{\delta}$. Suppose that the first segment connects p_0 to p_1 and the second one connects p_1 to p_2, as shown in Figure 3. Let L be the open segment of $(1/2, 0)$ that maps to the union of these two line segments. We leave off the two boundary points of L. Let σ_2 be the side of $\partial\Delta$ containing p_2. Let π_2 be a projection whose kernel is parallel to s_2. Then

Main Observation: *$\phi|_L$ is locally projective on L relative to π_2.*

The map $\widehat{\delta}$ is completely determined by its action on any neighborhood of $\{1/2\}$. Once we know this, we can uniquely continue $\widetilde{\delta}$ all the way to 0 just using the Main Observation.